УДК 519.68: 681.51 512.573

GREEN'S FUNCTION AND A SEMI-ANALYTICAL METHOD FOR THE CHANNEL TURBULENT FLOW[1]

Sumbatyan M. A.[2], Ricci Fabrizio[3], Vaccaro Massimo[4]

ФУНКЦИЯ ГРИНА И ПОЛУАНАЛИТИЧЕСКИЙ МЕТОД ДЛЯ ТУРБУЛЕНТНОГО ТЕЧЕНИЯ В КАНАЛЕ

Сумбатян М. А., Риччи Ф., Ваккаро М.

Предлагается полуаналитический метод решения классической задачи динамики вязкой жидкости о течении турбулентного однородного потока в канале постоянной ширины (двумерная задача). В стандартной итерационной трактовке пошагового движения вдоль временной переменной на каждом шаге итераций получается некоторая линейная эллиптическая задача четвертого порядка в полосе. В данной работе строится явное решение этой задачи в квадратурах. Для этого вначале с использованием интегрального преобразования Фурье вдоль канала строится функция Грина, удовлетворяющая необходимым граничным условиям для функции тока на стенках канала. Затем решение всей задачи выписывается в явном виде в терминах этой функции Грина.

1. Let us study the classical problem about a turbulent flow of the incompressible fluid in a channel of constant width (two-dimensional case). The Navier-Stokes equations written in terms of "vorticity – stream function" have the following form [1, 2]

$$\frac{\partial\zeta}{\partial t} = -u\,\frac{\partial\zeta}{\partial x} - v\,\frac{\partial\zeta}{\partial y} + \nu\Delta\zeta\,, \quad \zeta = \Delta\psi\,. \qquad (1)$$

The components of the velocity vector of any fluid particle $u(x,y,t)$, $v(x,y,t)$ are related with the stream function $\psi(x,y,t)$ and the vorticity $\zeta(x,y,t)$ in the following way

$$u = \frac{\partial\psi}{\partial y}\,, \quad v = -\frac{\partial\psi}{\partial x}\,, \quad \zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\,. \qquad (2)$$

To be more specific, let us assume that x coordinate is directed along the channel $(-\infty < x < \infty)$, and the transverse vertical coordinate varies over the interval $-h < y < h$. The width of the channel is thus $2h$.

If the average expense of the fluid is known

$$Q = \int\limits_{-h}^{h} u(x,y)dy = \psi|_{y=h} - \psi|_{y=-h}\,, \qquad (3)$$

being free of x and t, then the mean velocity of the flow $U_m = Q/(2h)$ is a known constant quantity. Besides, if we extract from the full solution a simple structure providing the given expense and no-slip boundary condition, then we can rewrite equations (1)–(2) for new functions satisfying the homogeneous conditions on the channel walls

$$\psi = \psi^0 + \frac{3}{4}\,Q\left(\frac{y}{h} - \frac{y^3}{3h^3}\right)\,,$$
$$\zeta = \zeta^0 - \frac{3}{2}\,\frac{Q\,y}{h^3}\,, \quad \frac{\partial\zeta}{\partial x} = \frac{\partial\zeta^0}{\partial x}\,,$$
$$\frac{\partial\zeta}{\partial y} = \frac{\partial\zeta^0}{\partial y} - \frac{3Q}{2h^3}\,,$$
$$u = u^0 + \frac{3Q}{4h}\left(1 - \frac{y^2}{h^2}\right)\,, \quad v = v^0\,, \qquad (4)$$
$$\frac{\partial\Delta\psi^0}{\partial t} - \nu\Delta^2\psi^0 = f\,,$$

---

[1]Работа выполнена при поддержке INTAS (04-80-7043).

[2]Сумбатян Межлум Альбертович, д-р физ.-мат. наук, профессор кафедры теоретической гидроаэромеханики Южного федерального университета (Ростовского государственного университета).

[3]Риччи Фабрицио, исследователь факультета аэронавтики Неапольского университета им. Федерико II, Италия.

[4]Ваккаро Массимо, исследователь Департамента прикладной математики Салернского университета, Италия.

$$\psi|_{y=\pm h} = \left.\frac{\partial\psi}{\partial y}\right|_{y=\pm h} = 0\,,$$

$$f = -\left[u^0 + \frac{3Q}{4h}\left(1 - \frac{y^2}{h^2}\right)\right]\frac{\partial\zeta^0}{\partial x} - \\ - v^0\left[\frac{\partial\zeta^0}{\partial y} - \frac{3Q}{2h^3}\right].$$

For large Reynolds numbers $Re = hU_m/\nu$ the flow is turbulent, and precise simulation of the flow providing correct calculation of the turbulent components of the velocity vector becomes complicated. In the meantime, the information about oscillating components of the velocity is of high importance, for example, in such fields as aeroacoustics [3]. That is why a number of direct numerical methods have been applied to solve nonlinear system of differential equations (1)–(2), — the methods efficient both in laminar and turbulent ranges (the reviews of the published works can be found in [1, 2, 4–8]).

One of known direct numerical methods is the method based on classical iterations in time. At each iteration step this is reduced to a linear elliptic problem over spatial variables. The main goal of the present work is to develop an explicit solution to this elliptic problem for the channel of constant width. The method to construct such a solution is founded on application of Green's function, which is also developed explicitly in quadratures.

2. The considered problem (1)–(2) on variable t is a classical Cauchy problem, if one assumes the initial conditions to be known. Usually in the developed stream such conditions are rarely known *a priori*, however numerical experiments show that the choice of these or those initial conditions does not affect the qualitative character of the flow. It is known (see, for instance, [1, 2]) that a stable numerical solution can be constructed by some iteration processes in time, if one uses an implicit finite-difference scheme in time (the so-called derivative "backwards"). The simplest scheme is the Euler one: $(\partial\zeta^0/\partial t)_n \approx (\zeta_n^0 - \zeta_{n-1}^0)/\tau$, where τ is the step in time. The substitution of this relation to Eq. (4) with function f containing nonlinear terms on the previous time layer and the term containing the higher fourth-order derivatives of function ψ — on the new time layer, reduces the problem to an elliptic boundary value problem of the fourth order regarding the stream function

$$\Delta\psi_n^0 - \nu\tau\Delta^2\psi_n^0 = g_{n-1}\,, \\ g_{n-1} = \zeta_{n-1}^0 + \tau\, f_{n-1}\,. \tag{5}$$

This scheme is linearly convergent, both in time and space, with the step τ decreasing. The second-degree convergence with respect to τ can be achieved by using Crank-Nicolson scheme instead of Euler's one (see, for example, [1]). This leads to an elliptic problem which is only slightly different from (5)

$$\Delta\psi_n^0 - \frac{\nu\tau}{2}\Delta^2\psi_n^0 = g_{n-1}^*\,, \\ g_{n-1}^* = \left(\zeta_{n-1}^0 + \frac{\nu\tau}{2}\Delta\zeta_{n-1}^0\right) + \tau\, f_{n-1}\,. \tag{6}$$

At last, second-degree convergence, both in time and space, can be attained if one applies the Crank-Nicolson scheme for the viscous term and the Adams-Bashforth scheme for the nonlinear term [6]

$$\Delta\psi_n^0 - \frac{\nu\tau}{2}\Delta^2\psi_n^0 = g_{n-1}^{**}\,, \tag{7}$$

$$g_{n-1}^{**} = \left(\zeta_{n-1}^0 + \frac{\nu\tau}{2}\Delta\zeta_{n-1}^0\right) + \\ + \frac{\tau}{2}\left(3f_{n-1} - f_{n-2}\right).$$

It is easily seen that all three versions of the iteration method can be written uniquely as follows

$$\Delta\psi_n^0 - \varepsilon\Delta^2\psi_n^0 = \hat{g}_{n-1}\,, \\ \psi|_{y=\pm h} = \left.\frac{\partial\psi}{\partial y}\right|_{y=\pm h} = 0\,, \tag{8}$$

where $\varepsilon > 0$ is a certain small parameter at highest-order derivatives, and $\hat{g}_{n-1}$ is a certain function known from the previous iterations.

3. In the case of the channel of constant width equation (8) possesses an exact explicit solution. In order to construct it, let us first construct Green's function, i.e. the solution to the following boundary value problem

$$\Delta G - \varepsilon\Delta^2 G = \delta(\xi - x)\delta(\eta - y)\,, \\ G|_{\eta=\pm h} = \left.\frac{\partial G}{\partial \eta}\right|_{\eta=\pm h} = 0\,, \tag{9}$$

where $G = G(\xi, \eta, x, y)$, and the Laplace operator is applied with respect to variables (ξ, η).

Application of the Fourier transform with respect to variable ξ ($\xi \Rightarrow \alpha$) reduces equation (9) to a linear ordinary differential equation of the fourth order with constant coefficients (Fourier images are designated by tildes)

$$\varepsilon\frac{d^4\tilde{G}}{d\eta^4} - (1+2\varepsilon\alpha^2)\frac{d^2\tilde{G}}{d\eta^2} + (\alpha^2+\varepsilon\alpha^4)\tilde{G} = = -\,e^{i\alpha x}\delta(\eta-y)\,, \quad (10)$$

$$\tilde{G}|_{\eta=\pm h} = \left.\frac{d\tilde{G}}{d\eta}\right|_{\eta=\pm h} = 0\,.$$

A particular solution to nonhomogeneous equation (10) can easily be developed as a series by separation of variables

$$\tilde{G}_p = -\frac{e^{i\alpha x}}{h}\sum_{m=1}^{\infty}\left(\frac{1}{\beta_m^2+\alpha_1^2} - \frac{1}{\beta_m^2+\alpha_2^2}\right)\times \times \sin\frac{\pi m\bar{y}}{2h}\,\sin\frac{\pi m\bar{\eta}}{2h}\,, \quad (11)$$

$$\bar{y} = y+h\,, \quad \bar{\eta} = \eta + h\,,$$

$$\alpha_1 = \alpha\,, \quad \alpha_2 = \sqrt{\alpha^2+\frac{1}{\varepsilon}}\,, \quad \beta_m = \frac{\pi m}{2h}\,,$$

which after summation of some table series can be rewritten in the form

$$\tilde{G}_p = -\frac{e^{i\alpha x}}{2}\times \times\left\{\frac{\mathrm{ch}[(2h-|\eta-y|)\alpha_1] - \mathrm{ch}[(\eta+y)\alpha_1]}{\alpha_1\,\mathrm{sh}(2h\alpha_1)} - \frac{\mathrm{ch}[(2h-|\eta-y|)\alpha_2] - \mathrm{ch}[(\eta+y)\alpha_2]}{\alpha_2\,\mathrm{sh}(2h\alpha_2)}\right\}. \quad (12)$$

If one adds to (12) a general solution of the homogeneous equation then one obtains for the general solution of equation (10) the following representation

$$\tilde{G} = \tilde{G}_p + C_1\mathrm{sh}[\alpha_1(\eta-h)] + C_2\mathrm{sh}[\alpha_1(\eta+h)] + + C_3\mathrm{sh}[\alpha_2(\eta-h)] + C_4\mathrm{sh}[\alpha_2(\eta+h)]\,. \quad (13)$$

The four unknown constants $C_1 - C_4$ should be determined from the four boundary conditions (10). Let us note that particular solution (11), (12) has been specially constructed so that the conditions $\tilde{G}_p(\eta = \pm h) = 0$ are automatically satisfied. Besides, a special choice of the structure of homogeneous equation leads to the 4×4 algebraic system regarding $C_1 - C_4$ such that four elements of the matrix among 16 elements are equal to zero. Besides, two elements among four ones in the right-hand side vector are equal to zero too. As a result, all four unknown coefficients are easily expressed in explicit form, and after some transformations the Fourier image of the Green's function can finally be represented as follows

$$\tilde{G}(\eta, x, y) = e^{i\alpha x}H;$$

$$H(\eta,y) = = \frac{\mathrm{ch}[(2h-|\eta-y|)\alpha_1] - \mathrm{ch}[(\eta+y)\alpha_1]}{2\alpha_1\,\mathrm{sh}(2h\alpha_1)} - \frac{\mathrm{ch}[(2h-|\eta-y|)\alpha_2] - \mathrm{ch}[(\eta+y)\alpha_2]}{2\alpha_2\,\mathrm{sh}(2h\alpha_2)} + + \{[\alpha_2\,\mathrm{sh}(2\alpha_1 h) - \alpha_1\,\mathrm{sh}(2\alpha_2 h)][G_0(y)G_0(-\eta) + + G_0(-y)G_0(\eta)] + + [\alpha_2\mathrm{sh}(2\alpha_1 h)\mathrm{ch}(2\alpha_2 h) - \alpha_1\mathrm{sh}(2\alpha_2 h)\mathrm{ch}(2\alpha_1 h)]\times \times [G_0(y)G_0(\eta) + + G_0(-y)G_0(-\eta)]\}\,\frac{1}{D_0}\,,$$

$$G_0(y) = \frac{\mathrm{sh}[(h+y)\alpha_1]}{\mathrm{sh}(2\alpha_1 h)} - \frac{\mathrm{sh}[(h+y)\alpha_2]}{\mathrm{sh}(2\alpha_2 h)}\,,$$

$$D_0 = (\alpha_1^2+\alpha_2^2)\mathrm{sh}(2\alpha_1 h)\mathrm{sh}(2\alpha_2 h) + + 2\alpha_1\alpha_2[1 - \mathrm{ch}(2\alpha_1 h)\mathrm{ch}(2\alpha_2 h)]\,. \quad (14)$$

Obviously, Green's function itself can be obtained by the inverse Fourier transform applied to Eq. (14)

$$G(\xi,\eta,x,y) = = \frac{1}{\pi}\int_0^\infty \cos[\alpha(x-\xi)]H(\eta,y)d\alpha\,. \quad (15)$$

As a result, exact solution to equation (8) is expressed in quadratures

$$\psi_n^0(x,y) = = \int_{-\infty}^{\infty}\int_{-h}^{h} G(\xi,\eta,x,y)\,\hat{g}_{n-1}(\xi,\eta)d\xi d\eta\,. \quad (16)$$

Conclusion. In the classical iteration processes in time, used to calculate turbulent flows in the channel of constant width, at each step of iterations there arises a liner elliptic boundary value problem of the fourth order with singular perturbations, containing a small factor at highest derivatives. In the present work this elliptic problem is solved in quadratures. This reduces the iteration process to calculation of integrals of some functions defined at previous steps of iteration.

It should be noted that the method proposed in the present paper can easily be extended to the three-dimensional problem for the channel of constant width. In this case the exact explicit solution of the elliptic boundary value problem, arising at each step in time, is constructed by the two-dimensional Fourier transform.

It should also be noted that the constructed Green's function permits reduction of the problem for arbitrary-shaped obstacle placed into the turbulent flow in the channel of constant width to a simple boundary integral equation.

The present work has been supported by INTAS, Grant 04-80-7043.

Литература

1. *Roache P. J.* Computational Fluid Dynamics. Hermosa Publ.: Albuquerque, 1976. 616 p.
2. *Fletcher C. A. J.* Computational Methods in Fluid Dynamics. Springer-Verlag: Berlin-Heidelberg, 1988. Vol. 1. 502 p. Vol. 2. 552 p.
3. *Goldstein M. E.* Aeroacoustics. McGraw-Hill: New York, 1976. 295 p.
4. *Peyret R., Taylor Th. D.* Computational Methods for Fluid Flow. Springer-Verlag: New York, 1983. 352 p.
5. *Moin P., Kim J.* Numerical investigation of turbulent channel flow // J. Fluid Mech. 1982. Vol. 118. P. 341–377.
6. *Kim J., Moin P., Moser R.* Turbulence statistics in fully developed channel flow at low Reynolds number // J. Fluid Mech. 1987. Vol. 177. P. 133–166.
7. *Rozhdestvensky B. L., Simakin I. N.* Modelling of turbulent flows in the plane channel // J. Comp. Math. Math. Phys. 1985. Vol. 25. No. 1. P. 96–121.
8. *Nikitin N. V.* Direct numerical modelling of three-dimensional turbulent flows in pipes of round cross-section // Mech. Fluids Gases. 1994. No. 6. P. 14–26.

Статья поступила 13 июня 2007 г.
Южный федеральный университет, г. Ростов-на-Дону
Неапольский университет, Италия
Салернский университет, Италия