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BLOCK ELEMENT METHOD FOR BODY, LOCALIZATIONS AND RESONANCES

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Abstract. For isotopic gibbous body deformed in a linear way differential and integral methods of factorization are applied. The algorithm of their usage is shown in details. It consists in submersion of boundary-value problem into topological area with topology induced by spheres of Euclidean space. The studying of the boundary-value problem is conducted in the space of slowly growing generalized functions. With application of means of external analysis the boundary-value problem is lead to the system of functional equations with matrix coefficient. The differential factorization that demanded building factorizing matrix-functions and automorphism allow obtaining pseudodifferential equation. For this purpose the Leray residue form is calculated. The integral equations meeting specific boundary conditions are extracted from it. The method is demonstrated by the example of boundary-value problem for the sphere, from which it is easy to see, that analytical notion of solution allows to reveal localizations and resonant parameter point simply enough.

Keywords: block element, factorization, spherical ball, integral and differential factorization methods, exterior forms, boundary problems.

Introduction

Let us assume that domain Ω occupied by an isotropic linearly deformable body in convex and its boundary $\partial\Omega$ is smooth. In the case of nonconvex boundaries, there are two ways to solve the boundary-value problem: either to pass on the generalized factorization or to subdivide the domain into block structures and investigate boundary-value problems in partial convex domains by means of simple factorization [1,2]. Note that the latter procedure implies that rectangular Cartesian coordinated are introduced at the tangent bundle of the boundary $\partial\Omega$ which is also used below. Consider the homogeneous differential Lamé equations in the conventional forms [1, 4]

$$(\lambda+\mu)\,\mathrm{graddiv}\,\mathbf{u}+\mu\Delta\mathbf{u}-\delta\mathbf{u}=0,\qquad \mathbf{u}=\{u_1,u_2,u_3\}\,. \tag{1}$$

Here, $\delta = -\rho\omega^2$ in vibration problems and $\delta=\rho p^2$ in nonstationary problems, where ω is a vibration frequency, p is the Laplace transform parameter, and ρ is the density of the material. In the boundary-value problem, certain boundary conditions to be set, which will be discussed below.

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1. The block element method

1. After applying a three-dimensional Fourier transform with operator $\mathbf{F}_3 = \mathbf{F}_3(\alpha_1, \alpha_2, \alpha_3)$ [1–4] over all the coordinates x_1, x_2, and x_2; substituting the $-i\alpha_k$ parameters of the Fourtier transform for the corresponding derivatives; and multiplying by -1, the above system of equations takes the following form:

$$\mathbf{KU} = \iint\limits_{\partial\,\mathbf{\Omega}} \boldsymbol{\omega}, \quad \mathbf{U} = \{U_1, U_2, U_3\}, \qquad (1.1)$$
$$\mathbf{U} = \mathbf{F}_3(\alpha_1, \alpha_2, \alpha_3)\mathbf{u},$$

$$\mathbf{K} = \begin{Vmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{Vmatrix},$$

$$k_{11} = (\lambda + 2\mu)\,\alpha_1^2 + \mu\alpha_2^2 + \mu\alpha_3^2 + \delta,$$
$$k_{12} = k_{21} = (\lambda + \mu)\,\alpha_1\alpha_2,$$
$$k_{13} = k_{31} = (\lambda + \mu)\,\alpha_1\alpha_3,$$
$$k_{22} = \mu\alpha_1^2 + (\lambda + 2\mu)\,\alpha_2^2 + \mu\alpha_3^2 + \delta,$$
$$k_{23} = k_{32} = (\lambda + \mu)\,\alpha_2\alpha_3,$$
$$k_{33} = \mu\alpha_1^2 + \mu\alpha_2^2 + (\lambda + 2\mu)\,\alpha_3^2 + \delta.$$

Let us consider a tangent of the boundary $\partial\Omega$ and introduce a local rectangular Cartesian coordinate system $\mathbf{x}^\nu$ such that the x_1^ν, x_2^ν axes lie in the tangent plane and the x_3^ν axis as aligned with the outward normal to boundary. The Fourtier transport parameters corresponding to them are denoted as α^ν. Formulas for the passage from one local system to another are given by the well-known transformation relationships:

$$\mathbf{x}^{\boldsymbol{\nu}} = \mathbf{c}_{\boldsymbol{\nu}}^{\boldsymbol{\tau}}\mathbf{x}^{\boldsymbol{\tau}} + \mathbf{x}_0^\tau, \quad \boldsymbol{\alpha}^{\boldsymbol{\nu}} = \mathbf{c}_{\boldsymbol{\nu}}^{\boldsymbol{\tau}}\boldsymbol{\alpha}^{\boldsymbol{\tau}}. \qquad (1.2)$$

Where $\mathbf{x}_0^{\boldsymbol{\tau}}$ are the coordinates of the origin of the new coordinate system in the initial one.

Using similar expressions, let us pass to new unknown quantities denied the following formulas:

$$\mathbf{u}^{\boldsymbol{\nu}} = \mathbf{c}_{\boldsymbol{\nu}}^{\boldsymbol{\tau}}\mathbf{u}^{\boldsymbol{\tau}}. \qquad (1.3)$$

Lemma. On the passage to the new local coordinate system, the images and preimages of the Fourtier transforms in Eqs. (1) are transformed according to formulas(1.2) and (1.3).

The Lemma is proved by direct substitution of the transform into (1), after which differential equations (1) should be written in each local coordinate system $\mathbf{x}^\nu$ with $\mathbf{u}^\nu = \{u_1^\nu, u_2^\nu, u_3^\nu\}$.

2. In functional equations (1.1), the vector of exterior forms ω has the following components [1–4]:

$$\omega_{sk} = R_{sk}dx_1\Lambda dx_2 + Q_{sk}dx_1\Lambda dx_3 + P_{sk}dx_2\Lambda dx_3, \qquad (1.4)$$

$$\boldsymbol{\omega} = \{\omega_{s1}, \omega_{s2}, \omega_{s3}\},$$

where subscript s indicates the group of exterior forms and k is the number of the row of the Lame equations. Transformation of the components $\mathbf{R}_3 = \{R_{31}, R_{32}, R_{33}\}$ of the obtained vector of the exterior form yields the following representation:

$$R_{31} = [\sigma_{13} - i\mu\alpha_3 u_1 - i\lambda\alpha_1 u_3]\, e^{i\langle\alpha x\rangle},$$
$$R_{32} = [\sigma_{23} - i\mu\alpha_3 u_2 - i\lambda\alpha_2 u_3]\, e^{i\langle\alpha x\rangle}, \qquad (1.5)$$
$$R_{33} = \left[\sigma_{33} - i\,(\lambda + 2\mu)\,\alpha_3 u_3 - i\mu\,(\alpha_1 u_1 + \alpha_2 u_2)\right] e^{i\langle\alpha x\rangle}.$$

Taking into account that an element of the tangent bundle is described by the oriented area $dx_1\Lambda dx_2$, we conclude that unknown quantities at the boundary can be set in terms of various combinations, using either stresses, or displacement, or mixed conditions.

Thus, for an isotropic body, the functional equations of the boundary-value problem under consideration in one of the local coordinate systems can be presented in the following form:

$$\mathbf{K}(\boldsymbol{\alpha}^\nu)\mathbf{U}^\nu = \iint\limits_{\partial\mathbf{\Omega}} \boldsymbol{\omega}^\nu = \sum_\tau \iint\limits_{\partial\mathbf{\Omega}} \varepsilon_\tau \boldsymbol{\omega}^\nu(\xi^\tau, \boldsymbol{\alpha}^\nu). \qquad (1.6)$$

where ε_τ is the partition of unity

3. For application of the differential factorization method to construction of the pseudodifferential equations for a matrix function, we use the approach developed in (1.5). As a result, the factorizing matrix functions take the following form:

$$\mathbf{Q}_1 = \begin{Vmatrix} \frac{1}{\alpha_3 - \alpha_{31-}} & \frac{-\alpha_1}{\alpha_2(\alpha_3 - \alpha_{31-})} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

$$\mathbf{Q}_2 = \begin{Vmatrix} \frac{1}{\alpha_3 - \alpha_{31-}} & 0 & \frac{-\alpha_2}{\alpha_{31-}(\alpha_3 - \alpha_{31-})} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix},$$

$$\mathbf{Q}_3 = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\alpha_1}{\alpha_{32-}(\alpha_3 - \alpha_{32-})} & \frac{\alpha_2}{\alpha_{32-}(\alpha_3 - \alpha_{32-})} & \frac{1}{\alpha_3 - \alpha_{32-}} \end{Vmatrix}.$$

Here, the double rots root of the determinant $\det\mathbf{K}$ are expressed as

$$\alpha_{31+} = i\sqrt{\alpha_2^2 + \alpha_1^2 + \frac{\delta}{\mu}},\ \alpha_{31-} = -\alpha_{31+}. \quad (1.7)$$

Where the signs at the subscrips indicate that the roots belong to the upper (plus) or lower (minus) half-planes of the complex plane.

The dimple roots are expressed as

$$\alpha_{32+} = i\sqrt{\alpha_2^2 + \alpha_1^2 + \frac{\delta}{\lambda+2\mu}}, \quad \alpha_{32-} = -\alpha_{32+}.$$

In order to obtain the required pseudodifferential equations, it is necessary to equate the corresponding Leray residue forms to zero. Calculating these residue forms in the neighborhood of the local coordinate system, we obtain the following relationships:

$$\begin{gathered}\lim_{\alpha_3\to\alpha_{31-}}(\alpha_3-\alpha_{31-})\mathbf{Q}_m\mathbf{F}_2\mathbf{R}_3 = 0,\\ m=1,2,\\ \lim_{\alpha_3\to\alpha_{32-}}(\alpha_3-\alpha_{32-})\mathbf{Q}_m\mathbf{F}_2\mathbf{R}_3 = 0,\\ m=3.\end{gathered} \quad (1.8)$$

Where $\mathbf{F}_2 = \mathbf{F}_2(\alpha_1,\alpha_2)$ is the two-dimensional Fourtier transform (with respect to the parameters x_1, x_2) of functions defined in the same neighborhood of the local coordinate systems.

In the matrix form, system (1.8) can be presented as

$$\begin{gathered}\mathbf{LF}_2\mathbf{u} = \mathbf{DF}_2\mathbf{t}, \quad \mathbf{t} = \{t_1,t_2,t_3\},\\ t_1=\sigma_{13}, \quad t_2=\sigma_{23}, \quad t_3=\sigma_{33},\end{gathered} \quad (1.9)$$

$$\mathbf{L} = \begin{Vmatrix} l_{11} & l_{12} & l_{13}\\ l_{21} & -l_{22} & 0\\ l_{31} & -l_{32} & -l_{33}\end{Vmatrix},$$

$$l_{11} = \alpha_1\sqrt{\tau_1^2-\upsilon^2},$$

$$l_{22} = \frac{1}{2}l_{33} = -\alpha_1\sqrt{\tau_2^2-\upsilon^2},$$

$$l_{12} = \alpha_2\sqrt{\tau_1^2-\upsilon^2}, \quad l_{21} = \alpha_2\sqrt{\tau_2^2-\upsilon^2},$$

$$l_{13} = s, \quad l_{31} = 2s+\alpha_2^2, \quad l_{32} = -\alpha_1\alpha_2,$$

$$\mathbf{D} = \frac{i}{\mu}\begin{Vmatrix} \frac{\alpha_1}{2} & \frac{\alpha_2}{2} & \frac{\sqrt{\tau_1^2-\upsilon^2}}{2}\\ \alpha_2 & -\alpha_1 & 0\\ \sqrt{\tau_2^2-\upsilon^2} & 0 & -\alpha_1\end{Vmatrix},$$

$$\tau_1 = -\frac{\delta}{\lambda+2\mu}, \quad \tau_2 = -\frac{\delta}{\mu},$$

$$\upsilon = \sqrt{\alpha_1^2+\alpha_2^2}, \quad s = 0.5\tau_2^2-\upsilon^2.$$

Calculation of the determinant of matrix L yields

$$\begin{gathered}\det \mathrm{L} = 2\alpha_1\sqrt{\tau_2^2-\upsilon^2}\times\\ \times\left[\upsilon^2\sqrt{\tau_1^2-\upsilon^2}\sqrt{\tau_2^2-\upsilon^2}+s^2\right] = \Delta_2,\end{gathered}$$

$$\alpha_{310} = -i\sigma_1 = \sqrt{\tau_1^2-\upsilon^2},$$

$$\alpha_{320} = -i\sigma_2 = \sqrt{\tau_2^2-\upsilon^2},$$

$$\operatorname{Im}\alpha_{3n0} \leqslant 0, \quad n=1,2.$$

Multiplying system (1.9) by the matrix-function L^{-1} on the left and applying the two-dimensional inverse Fourier transform, we get the following representation:

$$\mathbf{F}_2^{-1}\mathbf{K}_0\mathbf{F}_2\mathbf{t} = \mathbf{u}, \quad (1.10)$$

$$\mathbf{K}_0 = -\frac{1}{2\mu}\begin{Vmatrix} k_{11} & k_{12} & k_{13}\\ k_{21} & k_{22} & k_{23}\\ k_{31} & k_{32} & k_{33}\end{Vmatrix},$$

$$k_{11} = \alpha_1^2M+\alpha_2^2N, \quad k_{22} = \alpha_1^2N+\alpha_2^2M,$$

$$k_{12} = k_{21} = \alpha_1\alpha_2(M-N), \quad k_{33} = R,$$

$$k_{13} = -k_{31} = i\alpha_1P, \quad k_{23} = -k_{32} = i\alpha_2P,$$

$$M(\upsilon) = \frac{-0.5\tau_2^2\sigma_2}{\upsilon^2\Delta_0}, \quad N(\upsilon) = \frac{2}{\upsilon^2\sigma_2},$$

$$P(\upsilon) = \frac{\upsilon^2-0.5\tau_2^2-\sigma_1\sigma_2}{\Delta_0} \quad R(\upsilon) = \frac{-0.5\tau_2^2\sigma_1}{\Delta_0},$$

$$\Delta_0 = (\upsilon^2-0.5\tau_2^2)^2-\upsilon^2\sigma_1\sigma_2.$$

Using similar formulas, one can calculate the Leray reside forms in the right-hand side of functional equations (1.6) for remaining τ after the change of variables $\alpha^\tau = \mathbf{c}_\tau^\nu\alpha^\nu$.

An analysis of expression (1.10) shows that the obtained formulas coincide with those for the case where the body id half-space. However, it should be borne in mind that there is significant distinction consisting in the fact that functions u, t are defined in the neighborhood of the local coordinate systems generated by the tangents bundle of the boundary. Taking into account that the unity partition leads to coverage of the boundary by disjoint neighborhood, we conclude that the set of given and unknown functions for the system of pseudodifferential equations under consideration will contain functions defined in the neighborhoods of the local coordinate systems.

4. For further investigation, let us write the system of pseudodifferential equations (1.10) constructed after calculating the Leray residue forms as follows:

$$\iint\limits_{\partial\mathbf{\Omega}_\nu} \omega_0^\nu(\xi^\nu, \alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu)) +$$

$$+\sum_\tau{}' \iint\limits_{\partial\mathbf{\Omega}_\tau} \omega_0^\tau(\xi^\tau, \alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu)) = 0,$$

$$\nu = 1, 2, \ldots, T.$$

Here ω_0^ν, ω_0^τ are no longer the exterior forms; these quantities are given by expressions obtained after multiplying equations by the factorizing matrix functions and calculating the Leray residue forms. This system of equations can be rewritten in the following form:

$$\begin{aligned}
&\mathbf{L}^\nu(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu))\times\\
&\quad\times \mathbf{U}_0^\nu(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu)) -\\
&\quad- \mathbf{D}^\nu(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu))\times\\
&\quad\times \mathbf{T}^\nu(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu)) +\\
&\quad+\sum_{\tau=\mathbf{1}}^{\mathbf{T}}{}' \big[\mathbf{L}^\tau(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu))\times\\
&\quad\times \mathbf{U}_0^\tau(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu)) -\\
&\quad- \mathbf{D}^\tau(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu))\times\\
&\quad\times \mathbf{T}^\tau(\alpha_1^\nu, \alpha_2^\nu, \alpha_{3r-}^\nu(\alpha_1^\nu, \alpha_2^\nu))\big] = 0. \qquad (1.11)
\end{aligned}$$

Where the primed sum symbol implies that the term with $\tau = \nu$ in this sum is missing. The obtained pseudodifferential equations make it possible to formulate various boundary-value problems for elastic bodies. For example, let us assume that the displacement vector $\mathbf{u}^\nu$ is set at the boundary. Then, the system of equations can be rewritten as follows:

$$(\mathbf{L}^\nu)^{-1}\mathbf{D}^\nu\mathbf{T}^\nu + \sum_\tau (\mathbf{L}^\nu)^{-1}\mathbf{D}^\tau\mathbf{T}^\tau =$$

$$= \mathbf{U}_0^\nu + \sum_\tau (\mathbf{L}^\nu)^{-1}\mathbf{L}^\tau\mathbf{U}_0^\tau. \qquad (1.12)$$

We obtained the system of integral equations with respect to stresses, which can be written in a more explicit form by introducing, foe example, the following notation:

$$\mathbf{K}^\nu(\alpha_1^\nu, \alpha_2^\nu) = (\mathbf{L}^\nu)^{-1}\mathbf{D}^\nu,$$

$$\mathbf{K}^{\nu\tau}(\alpha_1^\nu, \alpha_2^\nu) = (\mathbf{L}^\nu)^{-1}\mathbf{D}^\tau,$$

$$\mathbf{B}^{\nu\tau}(\alpha_1^\nu, \alpha_2^\nu) = (\mathbf{L}^\nu)^{-1}\mathbf{L}^\tau.$$

As a result, applying the inverse Fourtier transform $\mathbf{F}_2^{-1}(x_1^\nu, x_2^\nu)$ with respect to parameters α_1^ν, α_2^ν, we arrive at the following system of integral equations:

$$\iint\limits_{\partial\mathbf{\Omega}_\nu} \mathbf{k}^\nu(x_1^\nu - \xi_1^\nu, x_2^\nu - \xi_2^\nu)\mathbf{t}^\nu(\xi_1^\nu, \xi_2^\nu)d\xi_1^\nu d\xi_2^\nu +$$

$$+\sum_{\tau=1}^{T}{}' \iint\limits_{\partial\mathbf{\Omega}_\tau} \mathbf{k}^{\nu\tau}(x_1^\nu, \xi_1^\tau, x_2^\nu, \xi_2^\tau)\mathbf{t}^\tau(\xi_1^\tau, \xi_2^\tau)d\xi_1^\tau d\xi_2^\tau =$$

$$= \mathbf{u}^\nu(x_1^\nu, x_2^\nu) +$$

$$+\sum_{\tau=1}^{T}{}' \iint\limits_{\partial\mathbf{\Omega}_\tau} \mathbf{b}^{\nu\tau}(x_1^\nu, \xi_1^\tau, x_2^\nu, \xi_2^\tau)\mathbf{u}^\tau(\xi_1^\tau, \xi_2^\tau)d\xi_1^\tau \xi_2^\tau, \qquad (1.13)$$

$$x_1^\nu, x_2^\nu \in \partial\mathbf{\Omega}_\nu, \quad 1 \leqslant \nu \leqslant T,$$

$$\mathbf{k}^\nu(x_1^\nu, x_2^\nu) = \mathbf{F}_2^{-1}\mathbf{K}^\nu(\alpha_1^\nu, \alpha_2^\nu),$$

$$\mathbf{k}^{\nu\tau}(x_1^\nu, \xi_1^\tau, x_2^\nu, \xi_2^\tau) = \\ = \mathbf{F}_2^{-1}\mathbf{K}^{\nu\tau}(\alpha_1^\nu, \alpha_2^\nu)\exp i\langle \mathbf{c}_\tau^\nu \alpha^\nu, \xi^\tau\rangle,$$

$$\mathbf{b}^{\nu\tau}(x_1^\nu, \xi_1^\tau, x_2^\nu, \xi_2^\tau) = \\ = \mathbf{F}_2^{-1}\mathbf{B}^{\nu\tau}(\alpha_1^\nu, \alpha_2^\nu)\exp i\langle \mathbf{c}_\tau^\nu \alpha^\nu, \xi^\tau\rangle.$$

where T is the number of local coordinate systems for the tangent bundle of the boundary. Similarly, one can derive the system of integral equations for a boundary-value problem with present stresses. The following theorem is valid.

Theorem. The operator $\mathbf{K}^\nu(\alpha_1^\nu, \alpha_2^\nu)$ in system (1.13) is principal, corresponding to the boundary-value problem for half-space; the remaining operators are subordinate, being completely continuous in spaces where yhe principal operator is invertible.

This theorem determines plenty of methods for the analytical and numerical investigation into system of integral equations of the type under consideration.

2. The ball body

We construct block elements with a spherical boundary by the differential factorization method. Contrary to the approaches described in [9–12], where simple factorization related to the representation of the group of translational motions of space was carried out, we applied generalized factorization [5] in this case. This approach is dictated by using the representation of the group of rotations of space induced by the sphere automorphism as a manifold with an edge. As in [9–12], we construct the functional and pseudo-differential equations for describing the block element as well as the representation of the solution for the boundary-value problem. Below, without repeating the general case [13], we presented the block elements for the ball and the space with the cutout ball and the Helmholtz equations derived for the boundary-value problems. The case under consideration is convenient because it makes possible to demonstrate the use of the method for problems solvable by other approaches. When using the method, we open distinctive features of the simple and generalized factorizations the application of the methods in an unlimited region, and the feature of satisfying the boundary conditions. The choice of the equation for the boundary-value problem is related also to the fact that the solutions of precisely this equation are the components of solutions of a number of boundary-value problems of a deformable-solid dynamic.

For an illustration, as an example, we constructed here the block elements for the boundary-value problem in the spherical region Ω_1 of radius b and in the space with the cutout spherical region Ω_2 of the radius a with the boundaries $\partial\Omega_s$, $s=1,2$, for the Helmholtz differential equation in the form of

$$\mathbf{Q}(\partial x_1, \partial x_2, \partial x_3)\varphi = \\ = \left[\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3 + k^2\right] \times \\ \times \psi(x_1, x_2, x_3) = 0. \quad (2.1)$$

It is shown below that the pseudo-differential equations for the block element enable us to consider all possible variants of boundary conditions θ, φ, r for the partial differential equation. For this purpose, we considered both the Dirichlet and Neumann boundary conditions as in the previous problems.

In the spherical system of coordinates θ, φ, r, Eq. (1) for the ball has the form

$$(\Delta + k_1^2)\psi = 0,$$

$$\Delta = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \\ + \frac{1}{r^2}\cdot\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \\ + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2}{\partial\varphi^2}, \quad (2.2)$$

$$r,\ \theta,\ \varphi \in \Omega_1.$$

A similar equation for the half-space with a cavity is taken in the form of

$$(\Delta + k_2^2)w = 0, \quad r,\ \theta,\ \varphi \in \Omega_2. \quad (2.3)$$

The solutions of the boundary-value problems for Eqs. (2.2), (2.3) are found in the spaces of slowly increasing generalized functions $\mathbf{H}_S$. For investigating this equation by the differential factorization method, we introduce the Fourier-Bessel transform and reversion in spherical functions of the form of

$$\mathbf{B}_2(l,m) = \\ = \int_0^\pi\int_0^{2\pi} g(\theta,\varphi)Y_l^{m-}(\theta,\varphi)\sin\theta d\theta d\varphi = G(l,m),$$

$$\mathbf{B}_2^{-1}(\theta,\varphi)G = \\ = \sum_{l=0}^\infty\sum_{m=-l}^{l} G(l,m)Y_l^{m+}(\theta,\varphi) = g(\theta,\varphi),$$

$$\mathbf{B}_3(\lambda,l,m)g = \\ = \int_0^\infty\int_0^\pi\int_0^{2\pi} g(r,\theta,\varphi)J_{l+\frac{1}{2}}(\lambda r)Y_l^{m-}(\theta,\varphi)\times \\ \times\sin\theta d\theta d\varphi r dr = G(\lambda,l,m), \quad (2.4)$$

$$\mathbf{B}_3^{-1}(r,\theta,\varphi)G = \\ = \sum_{l=0}^\infty\sum_{m=-l}^{l}\int_0^\infty G(\lambda,l,m)J_{l+\frac{1}{2}}(\lambda r)\times \\ \times Y_l^{m+}(\theta,\varphi)\lambda d\lambda = g(r,\theta,\varphi).$$

Here $J_\nu(\lambda r)$ is the Bessel function, and $Y_l^m(\theta,\psi)$ is the spherical function,

$$Y_l^{m\pm}(\theta,\varphi) = \\ = \frac{1}{2}\sqrt{\frac{2l+1}{\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{\pm im\varphi}.$$

Applying transforms (2.3) to Eq. (2.2), we construct the external form [14,15], which becomes (P, Q, R – some functions)

$$\omega = Pb^2\sin\theta d\theta\wedge d\varphi + \\ + Qbdr\wedge d\theta + Rb\sin\theta d\varphi\wedge dr. \quad (2.5)$$

We carry out the transition to the functional equation. It can be represented in the form [14, 15]

$$K(\lambda)\Psi(l,m,\lambda) = \int_{\partial\Omega}\omega, \quad K(\lambda) = \lambda^2 - k_1^2.$$

In the case of a ball, we have

$$(\lambda^2 - k^2)\Psi(l, m, \lambda) = L_{lm}(\lambda),$$

$$L_{lm}(\lambda) = b^2\psi'_{lm}(b)T_{lm}(\lambda, b) - b^2\psi_{lm}(b)T'_{lm}(\lambda, b), \quad (2.6)$$

$$\psi_{lm}(r) = \mathbf{B}_2(l, m)\psi(r, \theta, \varphi),$$

$$T_{lm}(\lambda, r) = \frac{1}{\sqrt{r}}J_{l+\frac{1}{2}}(\lambda r).$$

For providing the automorphism and obtaining the pseudo-differential equation, we construct the representation of the boundary-value problem solution as

$$\psi(r, \theta, \varphi) = \mathbf{B}_3^{-1}(r, \theta, \varphi)\frac{L_{lm}(\lambda)}{(\lambda^2 - k_1^2)}. \quad (2.7)$$

The automorphism requirement consists in fulfilling the equality [14, 15]

$$\psi(r, \theta, \varphi) = 0, \quad r > b. \quad (2.8)$$

As a result of simple transformations for the simple problem under consideration, we obtain a pseudo-differential equation degenerated into an algebraic one in the form of

$$L_{lm}(k_1) = 0. \quad (2.9)$$

In complex spatial problems, this equation is pseudo-differential literally.

By the example of this problem, it is already possible to observe the difference of the generalized factorization from the simple one: although the characteristic equation $K(\lambda)$ has two roots, Eq. (2.9) should be fulfilled only for one. A similar problem considered by simple factorization in a layer would require fulfilling Eq. (2.9) for both roots.

Using pseudo-differential Eq. (2.9), we consider the formulation of the boundary-value problems for Eq. (2.2). In the case of setting the Dirichlet conditions for the boundary $\partial\Omega$ for example, in the form of

$$\psi(b, \theta, \varphi) = \psi_0(b, \theta, \varphi). \quad (2.10)$$

The solution of pseudo-differential Eq. (2.9) is obtained in the form of

$$\psi'_{lm}(b) = \frac{\psi_{lm0}(b)T'_{lm}(k_1, b)}{T_{lm}(k_1, b)}.$$

Here we accept the designation

$$\psi_{lm0}(b) = \mathbf{B}_2(l, m)\psi_0(b, \theta, \varphi).$$

In the complex spatial problems, we obtain the integral or integro-differential equation instead of the algebraic one. For example, we applied the integral factorization method [16, 17] for its solution.

Introducing this relation into Eq. (2.7) and carrying out the necessary calculations, we obtain that, for $r \to b$, there takes place

$$\psi(r, \theta, \varphi) \to \psi_0(b, \theta, \varphi). \quad (2.11)$$

By using precisely the same algorithm, we solved the problem with the Neumann boundary condition. In this case, instead of boundary condition (2.10), the condition for the derivative is set on the boundary; i.e.,

$$\psi'(b, \theta, \varphi) = \psi_1(b, \theta, \varphi).$$

The solution of the pseudo-differential equation has the form of

$$\psi_{lm}(b) = \frac{\psi'_{lm1}(b)T_{lm}(k_1, b)}{T'_{lm}(k_1, b)}.$$

Here

$$\psi_{lm1}(b) = \mathbf{B}_2(l, m)\psi_1(b, \theta, \varphi).$$

Introducing this relation into Eq. (2.7) and carrying out the transformations, we again obtain the fulfillment of boundary conditions as in the Dirichlet problem; i.e.,

$$\psi'(r, \theta, \varphi) \to \psi_1(b, \theta, \varphi), \quad r \to b, \quad (2.12)$$

but for the classical component of the solution.

Conclusions

We particularly note that found solution (2.1) of the boundary-value problem in the space of slowly increasing generalized functions $\mathbf{H}_S$ consists of the classical component and the generalized functions. If the initial boundary-value problems are formulated for sufficiently smooth boundary conditions, for example, providing that the solutions belong to the Sobolev's spaces, the classical component coincides with this solution. The generalized component appears only as a result of the differentiation on the normal to the boundary $\partial\Omega$ of the step-function carrier. This circumstance is explained in detail in [1–4] and also in our subsequent studies.

Relation (2.12) is obtained with taking into account the remark made.

Thus, the pseudo-differential equation involves all possible variants of formulation of the boundary conditions of the problem.

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