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ON EQUILIBRIUM OF PENDANT DROP ITS FLEXURAL RIGIDITY OF INTERMEDIATE LAYER BEING ACCOUNTED FOR

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Abstract. We consider equilibrium of the axisymmetric drop pending from horizontal plane in the gravity field. Variational principle is formulated. It takes into account the energy necessary for the formation of the intermediate layer whose flexural rigidity is also considered. We prove existence of the solution of this problem and show that it is a classical solution of the nonlinear equation representing Euler condition for it.

Keywords: flexural rigidity, intermediate layer, variational principle, Laplace–Beltrami operator, mean and Gauss curvature, generalized derivatives, Sobolev spaces, weak convergence

1. Variational problem

In this paper we give in more details the results announced in the paper [1] of the authors. It is well known [2] that the Laplace condition determining the surface separating two phases in the state of equilibrium is not adequate when the surface curvature in one of the directions is sufficiently great. The possible cause is that it does not take into account the intermediate layer between two phases. In other words two phased system must be substituted by the three phased one. We can stay in the two phased model introducing into it the terms responsible for the influence of the intermediate layer. The models of such type were proposed in the papers [2, 3] (see also [4, 5]). In the paper [4] it was indicated that the flexural rigidity of the intermediate layer is also to be accounted for.

In our model we take into account all the above mentioned factors including into it the energy necessary for the formation of the intermediate layer and its flexural rigidity.

Let us denote by the letter S the axisymmetric surface of the drop and by the letter S^* – its projection onto horizontal plane P which it is pending from. We suppose that the line L generating the surface S is rectifiable and the functions

$$x = x(s), \quad y = y(s), \quad s \in (0, |L|), \quad y(0) \geqslant y_0$$

of its natural parametric representation possess generalized derivatives of the second order. We suppose also that

$$\dot{x}(s) \geqslant 0, \quad \dot{y}(s) \leqslant 0, \quad s \in (0, |L|)$$

Let Σ be the circle of the intersection of the surfaces S and S^*. Let us denote by the letter $\Im$ the class of admissible surfaces S we have just described.

In order to find the equilibrium forms of the pending drops with intermediate layer the variational principle for the functional

$$F = F(S) = \sigma \left(A(S) + l_p \Xi - \beta \int_S \mathrm{d}S + \lambda V + \sigma^{-1} \iiint_W \Gamma \rho \, \mathrm{d}V \right) \quad (1.1)$$

was formulated in the paper [6].

The coefficient σ is the coefficient of surface tension, the symbol $A(S)$ denotes the area of the surface of the drop, Γ – potential of the gravitational forces, ρ – density of the liquid, β – coefficient of adhesion, l_p — width of the intermediate layer and Ξ – functional representing energy necessary for the formation of the intermediate layer,

$$\Xi = \Xi(S) = 2\pi \int_0^{|L|} f(\dot{y}) \, \mathrm{d}s,$$

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}s},$$

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$$f = f(\dot{y}) = \frac{1}{2}\Bigg\{ -\sqrt{1-\dot{y}^2} \times \int_0^{|\dot{y}|} \left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2}\right) \times \left(1-\sigma^2\right)^{-\frac{3}{2}} d\sigma + E_0\sqrt{1-\dot{y}^2} \Bigg\} \quad (1.2)$$

The constant E_0 in the representation (1.2) may be selected in arbitrary way. We suppose that it is positive one.

On the basis of the molecular theory the energy of phase interaction on the separating surface in the two phased system was calculated in the paper [4]. The condition determining equilibrium surface and generalizing Laplace condition was also deduced there.

The condition includes the operator acting on mean and Gauss curvatures of the surface. It appears in the study of thin membranes by variational method. This was the reason for author of the paper to introduce the term of flexural rigidity of the separating surface. We also apply this term in our study of the equilibrium forms of the liquid drops. In our study of the equilibrium forms we propose the variational method permitting to take into account the flexural rigidity of the intermediate layer and the energy necessary for its formation. Thus in the class $\Im$ we consider the surfaces whose mean curvature H is square integrated

$$\|H\|_2^2 = \int_0^{|L|} H^2 \,\mathrm{d}S < \infty \quad (1.3)$$

Let V be the volume of the domain W bounded by the admissible surface and the plane P and V_0 positive number. We suppose that

$$V \geqslant V_0. \quad (1.4)$$

We denote by the symbol $\Im_H$ the subclass of the surfaces from the set $\Im$ satisfying the conditions (1.3), (1.4).

We study the variational problem for the following functional $F^* = F^*(S)$, $S \in \Im_H$

$$\begin{aligned} F^*(S) &= F(S) + \mu\sigma\|H\|_2^2 \\ &= \sigma A(S) + \sigma l_p \Xi - \beta\sigma \int_S \mathrm{d}S \\ &+ \lambda\sigma V + \iiint_W \Gamma\rho \,\mathrm{d}V + \mu\sigma\|H\|_2^2 \end{aligned} \quad (1.5)$$

Here as before $A = A(S)$ – the area of the surface S, Ξ – the functional described by the formula (1.2), (1.3).

Variational problem. For a given values of the non-negative constants V_0, l_p, β, λ, μ, σ, $\beta < 1$ and for the given non-negative and continuous functions Γ, ρ it is necessary to find the surface S_e from the class $\Im_H$ of admissible surfaces such that

$$F^*(S_e) = \inf\left\{F^*(S) \mid S \in \Im_H\right\} \quad (1.6)$$

Note. For the values $\mu = l_p = 0$ we get the classical variational problem in the class of the surfaces S generated by the rectifiable curves L ([7]).

2. Euler condition

First let us introduce the following definition.

Definition. Let $N = N(P)$ denotes the normal to the surface S at the point P. Let $P = P(s)$, $s \in [0, |\Sigma|]$, be parametric representation of the edge Σ of the surface S and $t = t(s) = t(P(s))$ – vector field of the tangent unit vectors along the line Σ. We'll call the vector product $\nu = \nu(s) = N(P(s)) \times t(P(s))$, $s \in |\Sigma|$, as the side normal to the surface S.

We'll prove now the following theorem.

Theorem 1. Let us suppose that there exists a solution $S_e \in \Im_H$ of the variational problem from the point 1 such that for each $\varepsilon > 0$ the functions x, y from the parametric representation of the curve L belong to the Sobolev space $W^{4,2}((0, |L| - \varepsilon))$. Let us suppose that the line L does not contain neither horizontal nor vertical segments. In this case the mean curvature H and Gauss curvature K of the surface $S_e \in \Im_H$ satisfy over $S_e \setminus \{(x(|L|), 0)\}$ the following condition

$$\mu\Delta_S H + 2\mu H\left(H^2 - K\right) + 2Hl_pK = \lambda + \frac{1}{\sigma}\Gamma\rho \quad (2.1)$$

Here Δ_S — Laplace–Beltrami operator defined over the surface S_e ([8,9]).

The contact angle γ between the surface S_e and horizontal plane the drop is pending from and the radius r of the circle S^* satisfy the following equations

$$\cos\gamma - \frac{l_p}{r}\left[\gamma - \frac{\sin 2\gamma}{2}\right] - \mu\frac{\partial H}{\partial\nu}\sin\gamma - \mu H^2\sin\gamma = \beta \quad (2.2)$$

$$\kappa V = \frac{1}{\pi r^2}\left[2\pi r\cos\gamma + \frac{l_p\pi r^2}{2}\sin^2\gamma - \pi r^2\int\limits_S\left[\mu\Delta_S H + 2\mu H\left(H^2 - K\right)\right]\mathrm{d}S\right] - \lambda \quad (2.3)$$

Here

$$\kappa = \frac{\rho g}{\sigma}$$

We get the classical result ([7]) when the coefficients l_p and μ are equal to zero.

Proof. In order to obtain (2.1)–(2.3) we write Euler necessary condition of the existence of the solution for the variational problem in the vicinities of the points lying on the surface S_e and on the side Σ. We direct displacements of the line L along the axis x in both of these cases. Let

$$x = x_1, \quad y_1 = y + \varepsilon t + \circ(\varepsilon), \quad \varepsilon \to 0 \quad (2.4)$$

Here $t = t(s)$ denotes infinitely differentiable function assuming non negative values and defined in the neighborhood of the point. $P \in S_e$. This neighborhood is unilateral when this point belongs to the side Σ. It is evident that the transformations of these kind does not diminish the volume of the domain bounded by the surface and horizontal plane.

Let L_1 be the curve obtained from the curve L by application of the transformation (2.4) to it. Let S_1 be the surface generated by the line L_1.

Let H_1 denotes the mean curvature of the surface S_1 and H – that of the surface S_e. Direct calculations show that the following equalities take place

$$H_1 = H + \frac{\varepsilon}{2}\left[\frac{\dot{x}t}{y^2} + \frac{\dot{x}\dot{y}\dot{t}}{y} + \frac{d\left(\dot{x}\dot{t}\right)}{\mathrm{d}s} + \ddot{x}\dot{t}\right] + \circ(\varepsilon), \quad \varepsilon \to 0 \quad (2.5)$$

Let s_1 be natural parameter of the curve L_1. For the variation of the differential $y\,\mathrm{d}s$ we get the following representation

$$y_1\mathrm{d}s_1 - y\,\mathrm{d}s = -2\varepsilon t\dot{x}yH\,\mathrm{d}s + \varepsilon\dot{\Omega}\,\mathrm{d}s + o(\varepsilon), \quad (2.6)$$
$$\varepsilon \to 0$$

The dot over the signal of the function means its differentiation in order of natural parameter and

$$\Omega = y\dot{y}t$$

Let

$$\bar{r} = \bar{r}(s,\varphi) = \left(x(s), \eta(s), \zeta(s)\right) = \left(x(s), y(s)\sin\varphi, y(s)\cos\varphi\right)$$

be the parametric representation of the axisymmetric surface S in the coordinate system $\{x, \eta, \varsigma\}$ of the three dimensional space and φ – the angle of the inclination of the meridian plane (x, y) to the plane (x, ζ). We denote as $\nabla_S f$ the gradient of the function f defined over the surface S,

$$\nabla_S f := g^{\alpha\beta} f_{,\beta}\bar{r}_\alpha$$

Here $\left(g^{\alpha\beta}\right)$ – matrix inverse to that of metric tensor represented by $g_{\alpha\beta}$, $\bar{r}_{,\alpha}$ – vector tangential to the surface in the direction α and $f_{,\beta}$ – the derivative of the function f in the direction of the axis β.

We have the following equality for the calculation of the gradient in the case of axisymmetrical surface

$$\nabla_S f = \left(\dot{x}\bar{i} + \dot{y}\sin\varphi\bar{j} + \dot{y}\cos\varphi\bar{k}\right)\frac{\partial f}{\partial s} + \frac{1}{y^2}\left(y\cos\varphi\bar{j} - y\sin\varphi\bar{k}\right)\frac{\partial f}{\partial\varphi} \quad (2.7)$$

It is easy to prove that the following representation ([8,9]) takes place for the Laplace–Beltrami operator Δ_S

$$\Delta_S f = \frac{\partial^2 f}{\partial s^2} + \frac{\dot{y}\dot{f}}{y} \quad (2.8)$$

Now using the formulas (2.7), (2.8) we can write down the function H_1 in the following form

$$H_1 = \left\{ H + \varepsilon \left[\frac{1}{2} \Delta_s (t\dot{x}) + (\nabla_S H \nabla_S \sqrt{g})\, t \right] + \varepsilon \left(2H^2 - K\right) \dot{x} t \right\} + \circ(\varepsilon), \quad \varepsilon \to 0 \quad (2.9)$$

Let $\delta_1 \|H\|_2^2 = \|H_1\|_2^2 - \|H\|_2^2$ be the variation of the functional $\|H\|_2^2$. Using now the formulas (2.6), (2.9) we arrive at the following expression

$$\delta_1 \|H\|_2^2 = 2\varepsilon\pi \int\limits_0^{|L|} \Delta_S H t \dot{x} y \, \mathrm{d}s + 2\pi\varepsilon \int\limits_0^{|L|} 2H \left(H^2 - K\right) \dot{x} y t \, \mathrm{d}s + \circ(\varepsilon), \quad \varepsilon \to 0 \quad (2.10)$$

Summing up the variations obtained for the different parts of the functional F^* we obtain that the variation $\delta_1 F^*$ of the functional F^* is equal to

$$\delta_1 F^* = \varepsilon\sigma \left\{ -2\pi \int\limits_0^{|L|} 2H y t \dot{x} \, \mathrm{d}s - 2\pi l_p \int\limits_0^{|L|} K y t \dot{x} \, \mathrm{d}s + 2\pi\lambda \int\limits_0^{|L|} y t \dot{x} \, \mathrm{d}s + \frac{1}{\sigma} 2\pi \int\limits_0^{|L|} \Gamma\rho t y \dot{x} \, \mathrm{d}s \right\} - \varepsilon 2\pi\mu\sigma \int\limits_0^{|L|} \left[\Delta_S H + 2H \left(H^2 - K\right)\right] t y \dot{x} \, \mathrm{d}s + \circ(\varepsilon), \quad \varepsilon \to 0 \quad (2.11)$$

Index 1 indicates that the variations are calculated in the interior part of the surface S_e.

The function t is an arbitrary one which means that the condition (2.1) is satisfied in the interior part of the surface S. The analogous calculations made on the boundary lead us to the formula (2.2). Integrating the expression from (2.1) over the surface S and taking into account the equality (2.2) we arrive at the equality (2.3).

The theorem is proved. □

3. Existence of variational solution

In this section we prove the following theorem.

Theorem 2. For a given values of the non-negative coefficients λ, μ, σ, l_p, β, $\beta < 1$ and for the given non-negative and continuous functions $\Gamma\rho$ there exists a solution $S_e \in \Im_H$ of the variational problem from the point 1 such that the functions x, y from the natural parametric representation of the line L generating it belong to the weighted Sobolev space $W^{4,2}\left((0, |L|), y\right)$. The mean and Gaussian curvatures of extreme surface $S_e \in \Im_H$ satisfy the condition (2.1), the contact angle γ and the radius r of the Σ – the conditions (2.2), (2.3) respectively.

Proof. First of all we note that the constant E_0 is at our disposal. It means that we can count that the values of the functional F are bounded from below by zero.

Let $\{S_n\}$ be minimizing sequence of the variational problem under consideration and $|L_n|$ – the length of the line L_n generating the surface S_n. It is clear that the sequence $\{|L_n|\}$ is the bounded one. Really the lengths $|L_n|$ could not tend to infinity because in this case the values of $F(S_n)$ should also tend to infinity which is impossible.

As the volumes of the domains corresponding to these surfaces are bounded from bellow we get that they do not tend to the trivial surface corresponding to the line consisting of segments of the axis. Let $z_n = x_n(s) + i y_n(s)$, $s \in [0, |L_n|]$ be the natural parametric representation of the curve L_n.

Let

$$X = \frac{x - y}{\sqrt{2}}, \quad Y = \frac{x + y}{\sqrt{2}}$$

be a new coordinate system obtained by performing rotation through $-45°$ of the coordinate system x, y. The derivatives of the first order of the functions x_n, y_n satisfy the following inequalities

$$\dot{x}_n(s) \geqslant 0, \quad \dot{y}_n(s) \leqslant 0, \quad s \in (0, |L|)$$

From these assumptions follows that there exist the functions $Y_n = Y_n(X)$ such that

$$x_n = \frac{\sqrt{2}}{2}\left[Y_n(X) + X\right],$$

$$y_n = \frac{\sqrt{2}}{2}\left[Y_n(X) - X\right].$$

These functions give non parametric representations of the curves L_n Extending if necessary the graphs of these functions along axis x and y we can suppose that all of these functions are defined over the same closed interval Δ. With easy calculations ([10]) we can show that the following inequalities are valid

$$\left|Y_n\left(X''\right) - Y_n\left(X'\right)\right| \leqslant \left|X'' - X'\right|.$$

For the extremal sequence $\{L_n\}$ all the terms on the left side of the last inequality are uniformly bounded.

Thus we get that the sequence $\{Y_n\}$ is compact in the sense of uniform convergence. We leave the notation of the sequence for the convergent subsequence. This means that the sequence $\{x_n\}$, $\{y_n\}$ are also convergent.

Now let us consider the convergence of the sequences $\{x_n\}$, $\{y_n\}$ of the higher order. To this end we prove the following lemma.

Lemma 3. The integrals

$$\int\limits_0^{|L_n|} \left(\ddot{x}_n^2 + \ddot{y}_n^2\right) y_n \, \mathrm{d}s$$

of the functions $\ddot{x}_n$, $\ddot{y}_n$ corresponding to the extremal sequence $\{L_n\}$ are uniformly bounded

Proof of the lemma. Really for each n the second order generalized derivative $\ddot{y}_n$ exists over $(0, |L_n|)$. It means that the functions $\dot{y}_n$, y_n, $\dot{y}_n$ are absolutely continuous in this interval. Thus we have

$$\int\limits_0^{|L_n|} \left(\ddot{x}_n^2 + \ddot{y}_n^2\right) y_n \, \mathrm{d}s$$
$$= \int\limits_0^{|L_n|} 4H_n^2 y_n \, \mathrm{d}s - \int\limits_0^{|L_n|} \dot{x}_n^2 \, \mathrm{d}s + \int\limits_0^{|L_n|} y_n \ddot{y}_n \, \mathrm{d}s$$
$$= \int\limits_0^{|L_n|} 4H_n^2 y_n \, \mathrm{d}s - \int\limits_0^{|L_n|} \left(\dot{x}_n^2 + \dot{y}_n^2\right) \mathrm{d}s$$
$$+ \int\limits_0^{|L_n|} \frac{\mathrm{d}(y_n \dot{y}_n)}{\mathrm{d}s} \leqslant \int\limits_0^{|L_n|} 4H_n^2 y_n \, \mathrm{d}s + |L_n| + y_n(0).$$

For the extremal sequence $\{L_n\}$ all the terms on the left side of the last inequality are uniformly bounded.

The lemma is proved. □

Now let us pass to the further investigation of the sequences $\{x_n\}$, $\{y_n\}$.

Now let us pass to the further investigation of the sequences $\{x_n\}$, $\{y_n\}$. It is clear that we can assume that the sequence $\{|L_n|\}$ is convergent one. Let $|L| := |L_n|$, $\lambda_n := |L_n| / |L|$ and let $\tilde{x}_n(s) := x_n(\lambda_n s)$, $\tilde{y}_n(s) := y_n(\lambda_n s)$, $s \in [0, |L|]$ be the new parametric representation of the curve L_n.

We show now that the sequences $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ converge uniformly over the interval $[0, |L| - \varepsilon]$. Indeed the functions of the sequences are absolutely continuous and for example

$$\left|\tilde{x}_n(s_1) - \tilde{x}_n(s_2)\right| \leqslant \lambda_n |s_1 - s_2|$$

The sequence $\{\lambda_n\}$ is bounded. In fact it converges to unity. Thus we get compactness of the sequences $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$.

From the lemma 3 it follows that for each $0 < \varepsilon < |L|$ the sequences $\{\tilde{x}_n\}$, $\{\{\tilde{y}_n\}\}$ are bounded in the Sobolev space $W^{2,2}((0, |L| - \varepsilon))$. Let us denote

$$x = x(s) = \lim_{n\to\infty} \tilde{x}_n(s),$$

$$y = y(s) = \lim_{n\to\infty} \tilde{y}_n(s),$$

and $y_\varepsilon = \min\{y(s), s \in [0, |L| - \varepsilon]\}$.

It is clear that for n sufficiently great the following estimate is valid

$$y_\varepsilon \int\limits_0^{|L|-\varepsilon} \left(\ddot{\tilde{x}}_n^2 + \ddot{\tilde{y}}_n^2\right) \mathrm{d}s \leqslant 2 \int\limits_0^{|L|} \tilde{y}_n \left(\ddot{\tilde{x}}_n^2 + \ddot{\tilde{y}}_n^2\right) \mathrm{d}s \leqslant 2\lambda_n^2 \int\limits_0^{|L_n|} y_n \left(\ddot{x}_n^2 + \ddot{y}_n^2\right) \mathrm{d}s$$

It is now clear that the sequences $\{\dot{\tilde{x}}_n\}$, $\{\dot{\tilde{y}}_n\}$ are uniformly convergent over each of the closed intervals $[0, |L| - \varepsilon]$, $0 < \varepsilon < |L|$. It is clear also that $z = z(s) = x(s) + iy(s)$, $s \in [0, |L|]$ is the natural parametric representation of the limit curve L for the sequence $\{L_n\}$ and $\dot{x}(s) \geqslant 0$, $\dot{y}(s) \leqslant 0$ over $(0, |L|)$. Besides the theorem of Banach–Alaoglu ([11]) guarantees that the sequences $\{\tilde{x}_n\}$, $\{\{\tilde{y}_n\}\}$ are compact in the sense of weak convergence in the space $W^{2,2}((0, |L| - \varepsilon))$. It means that the limit functions $x(s)$, $y(s)$ belong to this space.

In order to prove that the surface S_e generated by the line L belongs to the set $\Im_H$ it is left to show that for the mean curvature of the surface S_e the condition (1.4) is valid.

Due to weak convergence of $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$ in $W^{2,2}((0, |L| - \varepsilon))$ we have that

$$\int\limits_0^{|L|-\varepsilon} y \left(\ddot{x}^2 + \ddot{y}^2\right) \mathrm{d}s \leqslant 3 \int\limits_0^{|L_n|} y_n \left(\ddot{x}_n^2 + \ddot{y}_n^2\right) \mathrm{d}s$$

for each $0 < \varepsilon < |L|$. In the same manner we can evaluate the integral

$$\int\limits_S \frac{\dot{x}^2}{y^2} \, \mathrm{d}S$$

Thus we get that there exists a constant c_0 such that

$$\int\limits_S H^2 \, \mathrm{d}S \leqslant c_0 \int\limits_{S_n} H_n^2 \, \mathrm{d}S$$

These calculations show that the surface S_e belongs to the set $\Im_H$.

We'll show now that the surface S satisfies all the conditions of the theorem 2.

The direct calculations lead us to the following Euler condition for the extremal surface S

$$\int\limits_0^{|L|} \left[y\dot{y} + l_p \left(f_{\dot{y}} \dot{x}^2 + f(\dot{y})\, \dot{y}\right) + \mu 2y\ddot{x} + H^2 y\dot{y}\right] \dot{t} \, \mathrm{d}s + \int\limits_0^{|L|} \left(\lambda y\dot{x} + 1 + \kappa yx\dot{x} + \mu H \frac{\dot{x}}{y^2} + \mu H^2\right) t \, \mathrm{d}s + 2\mu \int\limits_0^{|L|} H\dot{x}y\ddot{t} \, \mathrm{d}s = 0 \quad (3.1)$$

We can rewrite the condition (3.1) in the following form

$$\int\limits_0^{|L|} \left(\lambda y\dot{x} + 1 + \kappa yx\dot{x} + \mu H \frac{\dot{x}}{y^2} + \mu H^2\right) t \, \mathrm{d}s + 2\mu \int\limits_0^{|L|} \left[H\dot{x}y - G\right] \ddot{t} \, \mathrm{d}s = 0 \quad (3.2)$$

The function G from this representation is defined by the following expression

$$G = G(s) := \int\limits_0^s \left[y\dot{y} + l_p \left(f_{\dot{y}} \dot{x}^2 + f(\dot{y})\, \dot{y}\right) + 2\mu y\ddot{x} + H^2 y\dot{y}\right] \mathrm{d}\sigma$$

The equality (3.2) means that the function $-\dot{x}^2 + y\ddot{y} - G(s)$ possesses the second order generalized derivative given by the formula

$$\left[1 + \lambda y\dot{x} + \kappa yx\dot{x} + \mu H \frac{\dot{x}}{y^2} + \mu H^2\right] (2\mu)^{-1}$$

Now using the equality

$$H\dot{x}y = -\dot{x}^2 + y\ddot{y}$$

we get that the function $y\ddot{y}$ has generalized derivative of the second order.

It means that the function $-\dot{x}^2 + y\ddot{y}$ of the real variable possesses absolutely continuous derivative of the first order on the set $(0, |L|)$ (in the multidimensional case this result is not valid in general).

We arrive at the conclusion that the function $\ddot{y}$ has weak derivative of the first order locally integrated with square over the interval $(0, |L|)$. In this case the function $y = y\,(x)$ has derivative of the third order absolutely continuous over the set $(0, |L|)$.

We prove now that the function $\ddot{x}$ possesses the weak derivative over the interval $(0, |L|)$. In order to prove it we'll use that the function $H\dot{x}y$ is differentiable. Integrating by parts the expression (3.2) we get

$$\int\limits_0^{|L|} \left[y\dot{y} + l_p\left(f_{\dot{y}}\dot{x}^2 + f(\dot{y})\,\dot{y}\right.\right. \\ \left.\left. + \mu 2y\ddot{x} + H^2 y\dot{y}\right)\right]\dot{t}\,\mathrm{d}s \\ - 2\mu \int\limits_0^{|L|} \left(-2\dot{x}\ddot{x} + \dot{y}\dddot{y}\right)\dot{t}\,\mathrm{d}s \\ + \int\limits_0^{|L|} \Big(\lambda y\dot{x} + 1 + \kappa y x\dot{x} \\ + \mu H\frac{\dot{x}}{y^2} + \mu H^2\Big) t\,\mathrm{d}s = 0 \quad (3.3)$$

From the formula (3.3) we obtain that the function defined by the following expression

$$y\dot{y} + l_p\left(f_{\dot{y}}\dot{x}^2 + f\,(\dot{y})\,\dot{y}\right) + \mu 2y\ddot{x} \\ + H^2 y\dot{y} - 2\mu\left(-2\dot{x}\dddot{x} + \dot{y}\dddot{y}\right) \quad (3.4)$$

has the generalized derivative equal to the following function

$$\lambda y\dot{x} + 1 + \kappa y x\dot{x} + \mu H\frac{\dot{x}}{y^2} + \mu H^2$$

It was already shown that all the functions in the formula (3.4) for the exception of the function $y\ddot{x}$ are differentiable in the weak sense. It means that the function $\ddot{x}$ is also differentiable in the same sense and $\dddot{x}$ is absolutely continuous on the segment $(0, |L|)$.

As the function $H\dot{x}y$ is twice differentiable then the function $y = y\,(s)$ has derivative of the fourth order. Integrating the expression (3.3) by parts and using the fact that the function $x = x\,(s)$ has derivative of the third order we get now that this function has derivative of the fourth order. Taking into account now the differentiable properties of the functions x, y we arrive at the following representation of the first variation of the functional $F^* = F^*\,(S)$

$$\delta_1 F^* = \varepsilon\sigma\Bigg\{-2\pi\int\limits_0^{|L|} 2Hyt\dot{x}ds \\ - 2\pi l_p \int\limits_0^{|L|} Kyt\dot{x}\,\mathrm{d}s + 2\pi\lambda\int\limits_0^{|L|} yt\dot{x}\,\mathrm{d}s \\ + \frac{1}{\sigma}2\pi\int\limits_0^{|L|}\Gamma\rho t\dot{x}\,\mathrm{d}s\Bigg\} \\ + 2\pi\mu\sigma\varepsilon\int\limits_0^{|L|}\Delta_S Ht\dot{x}y\,\mathrm{d}s \\ + 2\pi\mu\sigma\varepsilon\int\limits_0^{|L|} 2H\left(H^2 - K\right)\dot{x}yt\,\mathrm{d}s + \circ\,(\varepsilon)\,, \\ \varepsilon \to 0. \quad (3.5)$$

From formula (3.5) we now get that Euler condition can be written in the following form

$$\int\limits_0^{|L|}\left[y\dot{y} + l_p\left(f_{\dot{y}}\dot{x}^2 + f\,(\dot{y})\,\dot{y}\right)\right. \\ \left. + 2\mu y\ddot{x} + H^2 y\dot{y}\right]\dot{t}\,\mathrm{d}s \\ + \int\limits_0^{|L|}\left(\lambda y\dot{x} + 1 + \kappa y x\dot{x} + \mu H\frac{\dot{x}}{y^2} + \mu H^2\right) t\,\mathrm{d}s \\ + 2\mu\int\limits_0^{|L|} H\dot{x}y\ddot{t}\,\mathrm{d}s = 0 \quad (3.6)$$

From the formula (3.6) we obtain that the function $H\dot{x}y$ has the derivatives of the second order and the function $x = x\,(s)$, $y = y\,(s)$ – of the fourth order.

Let us now show that line L generating the surface S_e does not contain neither horizontal nor vertical segments.

Let us suppose that there is a vertical segment whose length is equal to h_1. Let us denote by the letter A the cylindrical part of the drop corresponding to this segment. Let us rearrange the drop substituting instead of the cylinder A the cylinder B of the same volume and whose base is on the plane P. Let r_1, r_2 be the radius of the bases of the cylinders A and B respectively. Then the following inequalities are valid

$$\pi r_1^2 = \pi r_2^2, \quad 2\pi r_2 h_2 < 2\pi r_1 h_1, \quad H_A > H_B$$

Here h_2 is the height of the cylinder B and H_A, H_B — their mean curvatures.

From the last inequalities it follows that

$$\sigma A\left(S\right) + \mu\sigma \left\|H\right\|_2^2$$

decreases under rearrangement. The term corresponding to the gravitational part of the functional does not increase. As for the other terms of the functional F they do not change under rearrangement.

Thus we get that the line L does not contain vertical segments.

In order to prove that it does not contain horizontal intervals it is sufficient to note that from the (3.3) it follows that the function $H$is absolutely continuous at the point of conjugation of this interval with the rest of curve. But it is equal to zero on this interval and is different from zero at the point of conjugation. We get a contradiction which means that the line L does not contain horizontal intervals.

From the formula (3.5) we get that the equality (2.1) is valid almost everywhere on S. In the usual way we now get the equalities (2.2) and (2.3) (see [7] and [6]).

The theorem is proved. □

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