Математика

УДК 517.5

ON FUNCTIONAL OF GAUSS CURVATURE

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Abstract. The authors study the problem of constructing a Gauss curvature functional whose variation on admissible surface is determined by its gauss curvature. In their previous papers the functional of such a type had been found for axisymmetrical surfaces. The crucial point in its constructing was correspondence between differential properties of the function determining it and and variation of the first quadratic form of the surface. Besides it was very important that axisymmetrical surfaces admit global half-geodesic parameterization. Thus, in order to obtain functional yielding gausss curvature in the general case it was necessary to revise differential equation whose solution gave Gauss curvature functoional in axisymmetrical case. This problemm was solved by analytic continution of the solution into upper half-plane. Further the problem of global half-geodesic parameterization arises. It is yet unsolved problem in the general setting. But the authors earlier showed that continuously differentiable surfaces with positively determined first quadratic form admit almost global half geodesic parameterization, i.e. it is possible to find a familly of geodesic lines covering it up to the null Hausdorff measure. The authors following the general lines study of axisysymmetrical case deduce integral-differential equations whose solution give desirable variation of the first quadratic form . This variation in accordance with differential properties of the solution of differentiable equation for the function determining gauss functional solve the problem of gauss curvature functional for the surfaces lacking axial symmetry.

Keywords: flexural rigidity of intermediate layer, capillar forces, Gauss curvature, mean curvature, Christoffel symbols, almost global half-geodesic parameterization, generalized analytic functions.

In the papers [1–3] of the authors were studied the equilibrium forms of the liquid drops pending from the flat surfaces in the axisymmetrical case. We use there variational method as a main tool of investigation. It was applied in this setting for the first time (compare with [4], see also [5, 6]).

In order to take into account intermediate layer we must substitute classical Laplace condition

$$\Delta p = \lambda H$$

by the following one

$$\Delta p = \lambda H + \theta K$$

Here Δp denotes difference of the pressures between liquid and gas phases, $\lambda\theta$ – positive constants characterizing capillary forces and those ones acting in the intermediate layer, H – mean curvature of the surface and K that of the Gauss.

Thus, the problem of the functional whose variation on the set of admissible functions is determined by Gauss curvature of surface arises. In the papers [1, 4] such a functional was introduced in the axisymmetrical case.

Let $x = x(s)$, $y = y(s)$, $s \in [0, L]$ be natural parametric representation of the twice-differentiable curve generating axisymmetrical surface S. Here (x, y) denotes Cartesian coordinates of the points in meridional section of the drop with axis x oriented along a line orthogonal to the plane from which the drop is pending.

The functional mentioned above and which we denote as Ξ has the following form

$$\Xi(S) = 2\pi \int\limits_0^{\dot{y}} f(\dot{y})\,\mathrm{d}s, \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}s} \tag{1}$$

Here function f

$$f(\dot{y}) = \frac{\sqrt{1-\dot{y}^2}}{2} \times \\ \times \left\{ E_0 - \int\limits_0^{\dot{y}} \left(\arcsin\sigma + \sigma\sqrt{1-\sigma^2} - \frac{\pi}{2} \right) \times \\ \times \left(\sqrt{1-\sigma^2} \right)^{-\frac{3}{2}} \right\} \mathrm{d}\sigma$$

is a solution of the differential equation

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$$\frac{\mathrm{d}^2 f}{\mathrm{d}\tau^2}\sqrt{1-\tau^2}-\frac{\mathrm{d}f}{\mathrm{d}\tau}\frac{\tau}{\sqrt{1-\tau^2}}+f\frac{1}{\sqrt{1-\tau^2}}=-1,$$

$$\tau\in[0,1]$$

We note here that functional Ξ is determined up to the constant by integrating function f along geodesics of surface S.

We are going now to generalize the functional (1) to the case of surfaces lacking axial symmetry. We consider now the set H of continuously differentiable non-parametric surfaces. We suppose that for each of these surfaces the function it representing has second generalized derivatives that locally integrates with square. We also suppose that they have plane of symmetry and intersection of this plane with the surface is its geodesic line of maximal length. It is clear that now we must restrict themselves to the surfaces, which admit covering by its geodesics at least up to the set of null Hausdorff measure. We proved ([7]) that such a covering is possible for surfaces of the set H.

Let

$$\mathrm{d}s^2=\mathrm{d}u^2+G\,\mathrm{d}v^2 \tag{2}$$

be the first quadratic form in half geodesic parameterization defined by the above – mentioned geodesic covering. In what follows we may assume that parametric domain is unit disk in the plane $w=u+iv$. It is well known that in this case as in axisymmetrical one Gauss curvature K may be represented in the form ([8])

$$K=-\frac{\ddot{\sqrt{G}}}{\sqrt{G}},\quad \ddot{\sqrt{G}}=\frac{\mathrm{d}^2\sqrt{G}}{\mathrm{d}s^2}$$

We assume further that the function $\ddot{G}$ is the bounded one.

In the general case, we are going to consider, the function $|\dot{G}|$ can assume the values greater than unit, which means that we cannot apply function (2) in this case.

The last obstacle we can easily overcome by introducing the following function $f^\star$ satisfying the following equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(1-t^2\right)\frac{\mathrm{d}f^*}{\mathrm{d}t}\right]+\frac{\mathrm{d}(tf^*)}{\mathrm{d}t}=-\sqrt{t^2-1} \tag{3}$$

We can write it down in an explicit form but it is not necessary to do. In the sequel, we use only the proper equation, which the function f^{**} satisfies. We get this equation using analytic continuation of the function f into upper half plane.

Let us consider on the space H the following functional of Gauss curvature

$$\Xi^*(S)=\int_{-1}^{1}\mathrm{d}v\int_{\sqrt{1-v^2}}^{\sqrt{1-v^2}}f^\star\left(\sqrt{\dot{G}}\right)\mathrm{d}u \tag{4}$$

$$\dot{G}=\frac{\mathrm{d}G}{\mathrm{d}s}=\frac{\mathrm{d}G}{\mathrm{d}u}$$

We are going to prove that Gauss curvature represents the linear part of the variation of the functional $\Xi^*(S)$ subjected to a special variation.

Let us consider surface S with non-parametric representation $z=z(x,y)$, $(x,y)\in D\subset R^2$. We do not discuss here the properties of boundary of domain D because we are interested in local variations of the surface S Let us consider in neighborhood N_0 of the point $(x_0,y_0)\in D$ variation S_ε of the surface S with non-parametric representation $z^\varepsilon=z^\varepsilon(x,y)$ of the following form

$$z^\varepsilon(x,y)=z(x,y)+\varepsilon[z\left(x+\alpha(x,y),y+\beta(x,y)\right)+\lambda\left(x,y\right)]+o(\varepsilon) \tag{5}$$

Here the functions α, β, λ are continuously differentiable and finite functions given in the neighborhood N_0 having generalized derivatives of the second order.

Theorem 1. Let

$$\mathrm{d}s_\varepsilon^2=\mathrm{d}u^2+G_\varepsilon\,\mathrm{d}v^2 \tag{6}$$

be first quadratic form of the surface $S_\varepsilon\in\mathfrak{H}$ corresponding to its almost global half-geodesic parameterization. We suppose that the second order-generalized derivatives of the function $z=z(x,y)$ are bounded in the neighborhood N_0 of the point $(x_0,y_0)\in D$, $\dot{x}$, $\dot{y}$ differ from zero and $\dot{\nabla}z\neq(0,0)$ in it. We suppose also that the function $\ddot{G}$ is bounded at the point (u_0,v_0), corresponding to the point (x_0,y_0) in half-geodesic parameterization. Then for any twice-differentiable λ there exist continuously-differentiable functions α, β belonging to the space $C(N_0)$ with bounded derivatives of the second order and such that

$$\sqrt{G_\varepsilon}=\sqrt{G}+\varepsilon\lambda+o\left(\varepsilon\right), \tag{7}$$

$$\mathrm{d}s_\varepsilon = \mathrm{d}s + \varepsilon\sqrt{G}\dot{\lambda}\,\mathrm{d}s + o(\varepsilon), \quad \varepsilon \to 0 \tag{8}$$

Proof. To begin with, we deduce for the functions α, β a system consisting of integral differential equation. We base our reasoning on assumption that equalities (7), (8) are valid. Solving the system, we get the functions α, β that satisfy theorem's conditions.

For the first, we find geodesics of the surface S_ε. Let

$$x = x(u, v_0), \quad y = y(u, v_0)$$

be parameterization of geodesic of the surface S and

$$\begin{aligned} x_\varepsilon = x_\varepsilon(u, v_0) &= x(u, v_0) \\ &+\varepsilon k(x(u, v_0), y(u, v_0)) + o(\varepsilon), \\ &\varepsilon \to 0, \\ y_\varepsilon = y_\varepsilon(u, v_0) &= y(u, v_0) \\ &+\varepsilon l(x(u, v_0), y(u, v_0)) + o(\varepsilon), \\ &\varepsilon \to 0 \end{aligned} \tag{9}$$

geodesics of the surface S_ε. We get from the equations (9) that

$$\begin{aligned} \frac{\mathrm{d}s_\varepsilon}{\mathrm{d}s} &= 1 + \varepsilon\frac{\mathrm{d}k^*}{\mathrm{d}u}\frac{\mathrm{d}x}{\mathrm{d}s} + \varepsilon\frac{\mathrm{d}l^*}{\mathrm{d}u}\frac{\mathrm{d}y}{\mathrm{d}s} \\ &+ \varepsilon\Lambda_x k^* + \varepsilon\Lambda_y l^* + o(\varepsilon), \quad \varepsilon \to 0 \end{aligned}$$

$$\begin{aligned} k^*(u) &:= k(x(u, v_0), y(u, v_0)), \\ l^*(u) &:= l(x(u, v_0), y(u, v_0)), \\ \Lambda_*(u) &:= \Lambda(x(u, v_0), y(u, v_0)) \end{aligned} \tag{10}$$

We see that functions kl determine variation of the arc length. We find them using formula (8) for it.

Let

$$\begin{aligned} x'' + \Gamma^1_{11}(x')^2 + 2\Gamma^1_{12}x'y' + \Gamma^1_{22}(y')^2 = 0, \\ y'' + \Gamma^2_{11}(y')^2 + 2\Gamma^2_{12}x'y' + \Gamma^2_{22}(y')^2 = 0 \end{aligned} \tag{11}$$

be equation of geodesic $x = x(u, v)$, $y = y(u, v)$ of the surface S corresponding to the fixed parameter v and

$$\begin{aligned} x''_\varepsilon + \Gamma^1_{11}(\varepsilon)(x'_\varepsilon)^2 + 2\Gamma^1_{12}(\varepsilon)x'_\varepsilon y'_\varepsilon \\ +\Gamma^1_{22}(\varepsilon)(y'_\varepsilon)^2 = 0, \\ y''_\varepsilon + \Gamma^2_{11}(\varepsilon)(x'_\varepsilon)^2 + 2\Gamma^2_{12}(\varepsilon)x'_\varepsilon y'_\varepsilon \\ +\Gamma^2_{22}(\varepsilon)(y'_\varepsilon)^2 = 0 \end{aligned} \tag{12}$$

be equation of geodesic $x_\varepsilon = x_\varepsilon(u, v)$, $y_\varepsilon = y_\varepsilon(u, v)$ of the surface S_ε corresponding to the same parameter.

Here Γ^k_{ij} – Christoffel symbols of the surface S and $\Gamma^k_{ij}(\varepsilon)$ – that of the surface S_ε.

Let $x := x_1$, $y := x_2$. Simple calculations show

$$\Gamma^k_{ij}(\varepsilon) = \Gamma^k_{ij} + \varepsilon\left(\gamma^k_{ij} + \widetilde{\gamma^k_{ij}}\right) + o(\varepsilon), \quad \varepsilon \to 0 \tag{13}$$

Here

$$\gamma^k_{ij} := \frac{z_{x_k}}{1 + |\nabla z|^2}\left\langle \Lambda^*, \frac{\partial}{\partial x_i}(\alpha, \beta)\right\rangle_{x_j}$$

$$\begin{aligned} \Lambda^\star := \big(&z_{x_1}(x_1 + \varepsilon\alpha(x_1,x_2), x_2 + \varepsilon\beta(x_1,x_2)), \\ &z_{x_2}(x_1 + \varepsilon\alpha(x_1,x_2), x_2 + \varepsilon\beta(x_1,x_2))\big) \end{aligned}$$

$$\widetilde{\gamma^k_{ij}} := \frac{\lambda_{x_j x_k} z_{x_k}}{1 + |\nabla z|^2} - 2R z_{x_k} z_{x_i x_j}$$

$$R := \frac{z^2_{x_1}\left\langle \Lambda^*, \frac{\partial}{\partial x_1}(\alpha, \beta)\right\rangle + z^2_{x_2}\left\langle \Lambda^*, \frac{\partial}{\partial x_2}(\alpha, \beta)\right\rangle}{\left(1 + |\nabla z|^2\right)^2} + \frac{\nabla\lambda \cdot \nabla z}{1 + |\nabla z|^2} \tag{14}$$

and $\langle e, g\rangle$ signifies scalar product of the vectors e, g.

Let now

$$b_k := \left(\gamma^k_{ij} + \tilde{\gamma}^k_{ij}\right)\left(\frac{\mathrm{d}x_1}{\mathrm{d}u}\right)^i\left(\frac{\mathrm{d}x_2}{\mathrm{d}u}\right)^j, \tag{15}$$

$$i, j = 1, 2,$$

$$\mathbf{b} = (b_1, b_2)\,h,$$

$$\mathbf{a} := \left(\frac{\mathrm{d}k}{\mathrm{d}u}, \frac{\mathrm{d}l}{\mathrm{d}u}\right)^T,$$

$$a_{ij} := \Gamma^i_{jl}\frac{\mathrm{d}x_l}{\mathrm{d}u}, \quad \mathbf{A} := (a_{ij})$$

Then from (11), (13), (14) we arrive at the boundary value problem for the following systems of ordinary differential equations on each of geodesic of the surface in the neighborhood W_0 of the point (u_0, v_0), corresponding to the point (x_0, y_0) in half-geodesic parameterization

$$\begin{aligned} \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}u} + A\mathbf{a} = \mathbf{b}, \\ \mathbf{a}(u_0 - \varepsilon) = \mathbf{a}(u_0 + \varepsilon) = 0. \end{aligned} \tag{16}$$

Taking into account differential properties of the function $z = z(x, y)$, we must limit our self by the generalized solutions of this equation. It means that in the neighborhood W_0 of the point (u_0, v_0) we procure function $\mathbf{a}$ that have uniformly bounded in W_0 generalized derivatives $\mathrm{d}\mathbf{a}/\mathrm{d}u$ on almost all the geodesics $\{v = \mathrm{const}\}$ from this neighborhood.

In order to find solutions of homogeneous system from (16) of this type we introduce the following integral equations

$$
\begin{aligned}
k(u,v) + \int\limits_{u_0-\varepsilon(v)}^{u} \left[a_{11}k(s,v) + a_{12}l(s,v)\right]\mathrm{d}s = 0, \\
l(u,v) + \int\limits_{u_0-\varepsilon(v)}^{u} \left[a_{21}k(s,v) + a_{22}l(s,v)\right]\mathrm{d}s = 0
\end{aligned}
\tag{17}
$$

in the space $C(W_0)$.

The systems (17) defines contraction operator in this space. Thus, there exists in this neighborhood measurable function ω_1 satisfying on almost all geodesics to differential equations (16) in the generalized sense and assuming zero values at the points $(u_0 - \varepsilon(v), v)$.

In the same manner, we find function ω_2 satisfying equations (16) and assuming zero value at the points $(u_0 + \varepsilon(v), v)$. Solutions we found are linearly independent.

In a standard way, we obtain now solutions $\mathbf{a}$ of the non-homogenous equation (16),

$$
\mathbf{a} = c_1(u,v)\,\omega_1 + c_2(u,v)\,\omega_2, \tag{18}
$$

$$
\omega_j = \left(\omega_j^1, \omega_j^2\right), \quad j = 1,2
$$

$$
c_1(u,v) = \int\limits_{u}^{u_0+\varepsilon} B_1\,\mathrm{d}\sigma, \quad c_2(u,v) = \int\limits_{u_0-\varepsilon}^{u} B_2\,\mathrm{d}\sigma
$$

$$
B_1 = \frac{b_1\omega_2^2 - b_2\omega_2^1}{\Delta_0}, \quad B_2 = \frac{b_2\omega_1^1 - b_2\omega_1^2}{\Delta_0}
$$

Using representation (18) we find the following functions

$$
k^*, \quad \frac{\mathrm{d}k^*}{\mathrm{d}u}, \quad l^*, \quad \frac{\mathrm{d}l^*}{\mathrm{d}u}
$$

Substituting them into formula (10) and using presupposed connection between $\mathrm{d}s$, $\mathrm{d}s_\varepsilon$, we arrive at the first integral-differential equation for the functions α, β

$$
\begin{aligned}
&(-1)^k\,\omega_k^j a_{ij}^* \frac{\mathrm{d}x^j}{\mathrm{d}s} B_k + (-1)^k \frac{\mathrm{d}\omega_k^j}{\mathrm{d}u} a_{ij}^* \frac{\mathrm{d}x^j}{\mathrm{d}s} \\
&\quad\times \int\limits_{u_0-(-1)^k\varepsilon}^{u} B_k\,\mathrm{d}\sigma + \Lambda_x k^* + \Lambda_y l^* + a_{kj}^* b_j \\
&\qquad = \sqrt{G}\dot{\lambda},
\end{aligned}
\tag{19}
$$

$$
k = 1,2
$$

Here $\left(a_{ij}^*\right)$ represents matrix inverse to the matrix $\mathbf{A}$.

The equations from (19) for the functions $\left\langle \Lambda^*, \frac{\partial}{\partial x_1}(\alpha,\beta)\right\rangle$, $\left\langle \Lambda^*, \frac{\partial}{\partial x_2}(\alpha,\beta)\right\rangle$ in fact are equivalent. In the sequel, we will use that one which corresponds to the value $k = 2$.

Substitute now instead of B_k, $k = 1,2$, their representations in terms of b_2. Then we obtain

$$
\begin{aligned}
&B^* b_2 + (-1)^k \frac{\mathrm{d}\omega_k^j}{\mathrm{d}u} a_{ij}^* \frac{\mathrm{d}x^j}{\mathrm{d}s} \\
&\quad\times \int\limits_{u_0-(-1)^k\varepsilon}^{u} B_k(b_2)\,d\sigma + \Lambda_x k^*(b_2) + \Lambda_y l^*(b_2) \\
&\qquad = \sqrt{G}\dot{\lambda}
\end{aligned}
$$

Here $k^*(b_2)$, $l^*(b_2)$ – operators of contraction whose norm can be made arbitrarily small. As the determinate Δ_0 can be made arbitrarily small coefficient B^* does not turn into zero.

Dividing by B^* we get integral equation with operator of contraction in the space $L^\infty(W_0)$ in the neighborhood (u_0, v_0). This permits us to find function b_2.

Let us deduce the second equation in order of the same unknowns. To this end, we notice that on the geodesics of the surfaces S, S_ε the following equations take place

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}s^2}\sqrt{G} + K\sqrt{G} = 0, \\
\frac{\mathrm{d}^2}{\mathrm{d}s_\varepsilon^2}\sqrt{G_\varepsilon} + K_\varepsilon\sqrt{G_\varepsilon} = 0
\end{aligned}
\tag{20}
$$

Here, the symbols K, K_ε denote Gauss curvatures of the surfaces S, S_ε respectfully.

Let us denote now

$$
\begin{aligned}
l_{ij} := \left\langle \Lambda^*, \frac{\partial}{\partial x_{(i+1)(mod2)}}(\alpha,\beta)\right\rangle_{x_{(j+1)(mod2)}} \\
+ \quad \lambda_{x_{(i+1)(mod2)}x_{(j+1)(mod2)}}
\end{aligned}
$$

Direct calculations show

$$K_\varepsilon = K - 2K\varepsilon \frac{z_{x_i}^2 \left\langle \Lambda^*, \frac{\partial}{\partial x_i}(\alpha,\beta)\right\rangle + \nabla\lambda\cdot\nabla z}{\left(1+|\nabla z|^2\right)^3} + (-1)^{i+j}\,\varepsilon \frac{z_{x_i x_j} l_{ij}}{\left(1+|\nabla z|^2\right)^2} + o(\varepsilon)\,, \quad \varepsilon\to 0 \tag{21}$$

Again we are summing up in (21) by repeating indexes.

We obtain further

$$\frac{\mathrm{d}^2}{\mathrm{d}s_\varepsilon^2}\sqrt{G_\varepsilon} = \ddot{\sqrt{G}} - \varepsilon\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\dot{G}\dot{\lambda}\right) - \frac{\lambda}{2\sqrt{G}}\right] + \frac{\mathrm{d}\dot{\sqrt{G}}}{\mathrm{d}s}\dot{\sqrt{G}}\dot{\lambda}\right\} + o(\varepsilon)\,, \quad \varepsilon\to 0 \tag{22}$$

Besides, we get

$$K_\varepsilon\sqrt{G_\varepsilon} = K\sqrt{G} + \varepsilon\frac{\lambda}{2\sqrt{G}} + (K_\varepsilon - K)\sqrt{G} + o(\varepsilon)\,, \quad \varepsilon\to 0 \tag{23}$$

Substituting now (22) and (23) into the second equation from (20) and taking into account (21) we arrive at the second equation we need

$$-2K(-1)^{i+j}\frac{z_{x_i x_j} l_{ij}}{\left(1+|\nabla z|^2\right)} + 2K\frac{z_{x_i}^2\left\langle \Lambda^*, \frac{\partial}{\partial x_i}(\alpha,\beta)\right\rangle + \nabla\lambda\cdot\nabla z}{\left(1+|\nabla z|^2\right)^3} = \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\dot{G}\dot{\lambda}\right) - \frac{\lambda}{2\sqrt{G}}\right] + \ddot{\sqrt{G}}\dot{\sqrt{G}}\dot{\lambda}' - K\frac{\lambda}{2\sqrt{G}} \tag{24}$$

This is the second equation we wished to get. It includes partial derivatives of the functions α, β.

It permits us to find differential equation for the functions

$$\left\langle \Lambda^*, \frac{\partial}{\partial x_i}(\alpha,\beta)\right\rangle, \quad j = 1,2.$$

Let

$$U := \left\langle \Lambda^*, \frac{\partial}{\partial x_1}(\alpha,\beta)\right\rangle, \qquad V := \left\langle \Lambda^*, \frac{\partial}{\partial x_2}(\alpha,\beta)\right\rangle$$

Then we get from (15)

$$U_{x_1}\left(\frac{\mathrm{d}x_1}{\mathrm{d}u}\right)^2 + \frac{\mathrm{d}x_1}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}\left[V_{x_1} + U_{x_2}\right] + V_{x_2}\left(\frac{\mathrm{d}x_2}{\mathrm{d}u}\right)^2 - 2\frac{z_{x_i x_j}}{\left(1+|\nabla z|^2\right)^2}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}\left(z_{x_1}^2 U + z_{x_2}^2 V\right) = b_2^\star \tag{25}$$

$$b_2^\star = b_2 + 2\frac{\nabla\lambda\cdot\nabla z}{\left(1+|\nabla z|^2\right)^2} z_{x_i x_j}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u} - \lambda_{x_i x_j}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}$$

In the same manner, we rewrite the equation (24)

$$-2K\frac{z_{x_2 x_2}}{\left(1+|\nabla z|^2\right)^2}U_{x_1} + 2K\frac{z_{x_1 x_2}}{\left(1+|\nabla z|^2\right)^2}\left[V_{x_1} + U_{x_2}\right] - U_{x_2} - 2K\frac{z_{x_1 x_1}}{\left(1+|\nabla z|^2\right)^2}V_{x_2} + 2K\frac{z_{x_1}^2}{\left(1+|\nabla z|^2\right)^3}U + 2K\frac{z_{x_2}^2}{\left(1+|\nabla z|^2\right)^3}V = g \tag{26}$$

$$g = -2K\frac{\nabla\lambda\cdot\nabla z}{\left(1+|\nabla z|^2\right)^3} + \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\dot{G}\dot{\lambda}\right) - \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\lambda}{2\sqrt{G}}\right)\right] + \frac{\mathrm{d}^2\sqrt{G}}{\mathrm{d}s^2}\dot{\sqrt{G}}\dot{\lambda}.$$

It is necessary to subline that in constructing the equation (26) we have substituted the function $\Lambda^\star$ by the function Λ this is possible because the solutions of the linear equations we are considering depend continuously on their coefficients.

At the same time, we are interested in the distortions of the initial surface up to the increments of the first order.

From the equations (25), (26) we now get

$$\frac{\mathrm{d}z_{x_2}}{\mathrm{d}u}\frac{\mathrm{d}x_1}{\mathrm{d}u}U_{x_1}+\frac{\mathrm{d}z_{x_1}}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}V_{x_2} - \left(z_{x_1x_2}+1\right)\frac{z_{x_ix_j}}{\left(1+|\nabla z|^2\right)^2} \times \frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}\left(z_{x_1}^2U+z_{x_2}^2V\right) = b_2^{\star}z_{x_1x_2}-g\frac{\mathrm{d}z_{x_1}}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}\left(\frac{2K}{\left(1+|\nabla z|^2\right)^3}\right)^{-1} =: b_2^{\star\star} \quad (27)$$

$$U_{x_1}\left(\frac{\mathrm{d}x_1}{\mathrm{d}u}\right)^2+\frac{\mathrm{d}x_1}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}\left[V_{x_1}+U_{x_2}\right] + V_{x_2}\left(\frac{\mathrm{d}x_2}{\mathrm{d}u}\right)^2 - 2\frac{z_{x_ix_j}}{\left(1+|\nabla z|^2\right)^2}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}\left(z_{x_1}^2U+z_{x_2}^2V\right) = b_2^{\star} \quad (28)$$

Let

$$\mathbf{M}_1=\begin{pmatrix}\frac{\mathrm{d}z_{x_2}}{\mathrm{d}u}\frac{\mathrm{d}x_1}{\mathrm{d}u} & \frac{\mathrm{d}z_{x_1}}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}\\ \left(\frac{\mathrm{d}x_1}{\mathrm{d}u}\right)^2 & \left(\frac{\mathrm{d}x_2}{\mathrm{d}u}\right)^2\end{pmatrix}$$

be matrix of the coefficients of the unknown U_{x_1}, V_{x_2} and $\kappa=\det\mathbf{M}_1$.

Solving (27), (28) in order of the functions U_{x_1}, V_{x_2}we get after simple transformations

$$U_{x_1}-V_{x_2}+\left(\frac{\mathrm{d}z_{x_1}}{\mathrm{d}u}\frac{\mathrm{d}x_2}{\mathrm{d}u}+\frac{\mathrm{d}z_{x_2}}{\mathrm{d}u}\frac{\mathrm{d}x_1}{\mathrm{d}u}\right)(\kappa)^{-1} \times\left(V_{x_1}+U_{x_2}\right)+L\left(U.V\right)=b_3$$

Here $L\left(U.V\right)$ is the first coordinate of the vector

$$\left(\mathbf{M}_1\right)^{-1}\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}$$

$$\Psi_1=-\left(z_{x_1x_2}+1\right) \times\frac{z_{x_ix_j}}{\left(1+|\nabla z|^2\right)^2}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}\left(z_{x_1}^2U+z_{x_2}^2V\right),$$

$$\Psi_2=-2\frac{z_{x_ix_j}}{\left(1+|\nabla z|^2\right)^2}\frac{\mathrm{d}x_i}{\mathrm{d}u}\frac{\mathrm{d}x_j}{\mathrm{d}u}\left(z_{x_1}^2U+z_{x_2}^2V\right)$$

and b_3 of the vector

$$\left(\mathbf{M}_1\right)^{-1}\begin{pmatrix}b_2^{\star\star}\\ b_2^{\star}\end{pmatrix}$$

Let now

$$\frac{\partial}{\partial\overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

In what follow we also will use the following notations

$$U_{\overline{z}}=\frac{\partial U}{\partial\overline{z}},\quad U_z=\frac{\partial U}{\partial z},$$

$$V_{\overline{z}}=\frac{\partial V}{\partial\overline{z}},\quad V_z=\frac{\partial V}{\partial z}$$

After simple calculations we now get

$$\left(\kappa^*+3i\right)W_{\overline{z}}-\left(\kappa^*+i\right)\overline{W}_{\overline{z}}+iL\left(U.V\right)=ib_3$$

Comparing this equation with the conjugated one

$$-\left(\kappa^*-i\right)W_{\overline{z}}+\left(\kappa^*-3i\right)\overline{W}_{\overline{z}}-iL\left(U.V\right)=-ib_3$$

we arrive at the generalized analytic functions equation

$$8W_{\overline{z}}+2i\left(\kappa^*-i\right)L\left(U.V\right)=-4ib_3 \quad (29)$$

Using Vekua operator we pass for the equation (29) is equivalent to the following integral equation

$$W\left(z\right)-\frac{1}{\pi}\iint\limits_{N_0}\frac{2i\left(\kappa^*-i\right)L\left(W.\overline{W}\right)\left(\zeta\right)}{\zeta-z}\mathrm{d}\xi\,\mathrm{d}\eta = \frac{1}{\pi}\iint\limits_{N_0}\frac{4ib_3}{\zeta-z}\mathrm{d}\xi\,\mathrm{d}\eta+\Phi\left(z\right), \quad (30)$$

$$\zeta=\xi+i\eta$$

Here $\Phi(z)$ is an arbitrary analytic function ([9]).

The equation (30) is an integral equation with operator of contraction for the sufficiently small N_0 Thus we get the functions U, V satisfying equations (25), (26) which satisfies zero condition on the boundary of N_0.

Now let us consider the following equations

$$\begin{aligned} U &:= \left\langle \Lambda, \frac{\partial}{\partial x_1}(\alpha,\beta) \right\rangle, \\ V &:= \left\langle \Lambda, \frac{\partial}{\partial x_2}(\alpha,\beta) \right\rangle \end{aligned} \tag{31}$$

Let $g := \alpha + i\beta$. After elementary calculations we get from (31) the following equation

$$\begin{aligned} &\left[\left(\frac{-i}{z_{x_1}+i} - \frac{z_{x_1}}{z_{x_1}-i}\right) - 1\right]\frac{\partial g}{\partial \overline{z}} - \overline{\frac{\partial g}{\partial \overline{z}}} \\ &= \frac{1}{z_{x_1}+i}(U - iV) - \frac{1}{z_{x_1}-i}(U+iV) \end{aligned} \tag{32}$$

Introducing now the equation that conjugates the equation (32) we exclude

$$\overline{\frac{\partial g}{\partial \overline{z}}}$$

As a result, we get non-homogeneous Cauchy–Riemann equation

$$\frac{\partial g}{\partial \overline{z}} = M(U,V)$$

Here $M(U,V)$ is a linear form depending on variables U, V. It is known that a general solution of this equation may be represented in the form

$$g(z) = F(z) - \frac{1}{\pi}\iint\limits_{N_0} \frac{M(U,V)(\zeta)}{\zeta - z}\,\mathrm{d}\xi\,\mathrm{d}\eta \tag{33}$$

with an arbitrary analytic function $F(z)$

As the function represented by integral operator from (33) is evidently Holder continuous, than we can select $F(z)$ in such a way that $g(z)$ would be equal to zero on the boundary of N_0.

Thus, we get functions α, β we needed.

The theorem is proved. □

Now we can prove the following theorem.

Theorem 2. The first variation of the functional (4) over space of admissible functions under conditions of the theorem 1 has the following representation

$$\delta\Xi^*(S) = \varepsilon \iint\limits_S -K\lambda\,\mathrm{d}S$$

Proof. Let us consider the difference

$$\begin{aligned} &\int_{-1}^{1}\mathrm{d}v \int_{\sqrt{1-v^2}}^{\sqrt{1-v^2}} f^\star\left(\sqrt{\dot{G}_\varepsilon}\right)\frac{\mathrm{d}s_\varepsilon}{\mathrm{d}u}\,\mathrm{d}u \\ &\quad - \int_{-1}^{1}\mathrm{d}v \int_{\sqrt{1-v^2}}^{\sqrt{1-v^2}} f^\star\left(\sqrt{\dot{G}}\right)\mathrm{d}u = \\ &= \int_{-1}^{1}\mathrm{d}v \int_{\sqrt{1-v^2}}^{\sqrt{1-v^2}} f^\star\left(\sqrt{\dot{G}} + \varepsilon\dot{\lambda} + o(\varepsilon)\right) \\ &\quad \times \left(1 - \varepsilon\sqrt{\dot{G}}\dot{\lambda} + o(\varepsilon)\right)\mathrm{d}u - \\ &\quad - \int_{-1}^{1}\mathrm{d}v \int_{\sqrt{1-v^2}}^{\sqrt{1-v^2}} f^\star\left(\sqrt{\dot{G}}\right)\mathrm{d}u \end{aligned} \tag{34}$$

As the function $f^\star$ satisfies the equation (3), then from (34) we get

$$\delta\Xi^*(S) = \varepsilon \iint\limits_S -K\lambda\,\mathrm{d}S$$

The theorem is proved. □

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Статья поступила 7 октября 2017 г.