Механика

УДК 539.3

DOI: 10.31429/vestnik-17-4-6-13

BLOCK ELEMENT METHOD IN SOLVING VECTOR BOUNDARY PROBLEMS USING SCALAR

V. A. Babeshko[1,2], O. V. Evdokimova[2], O. M. Babeshko[1]

[1] Kuban State University, Krasnodar, 350040, Russia

[2] Southern Scientific Center, Russian Academy of Science, Rostov-on-Don, 344006, Russia

e-mail: babeshko41@mail.ru

Abstract. This paper presents for the first time a solution of a vector boundary value problem decomposed over packed block elements that are solutions of scalar boundary value problems in a non-classical domain. Solutions of a number of vector partial differential equations in continuum mechanics, electromagnetic phenomena, and field theory allow representations in the form of decompositions based on solutions of scalar equations. This approach is convenient for solving problems in the entire space. When solving boundary value problems, the difficulty of applying this approach is the difficulty of satisfying boundary conditions. In a number of classical fields, this can be done and exact solutions to boundary value problems can be obtained. These classic areas include the half-space, the ball, the cylinder, and some areas obtained from views of transformation groups spaces. However, for a number of important areas other than classical ones, such as wedge-shaped ones, this approach has not yet been able to build accurate solutions. In this paper, probably for the first time, this approach is used to construct an exact solution in the first quadrant of a plane boundary value problem of the second kind for dynamic Lame equations. The solution is compared with the obtained direct application of the block element method to the vector boundary value problem. It is known that the unbounded domain makes it not effective to use numerical methods in this boundary value problem. The solution is constructed using the block element method under arbitrary boundary conditions. This makes it possible to study different properties of solutions by changing the effects on the boundary.

Keywords: boundary value problems, block element method, packed block elements, Lame and Helmholtz equations.

Introduction

It is known that the construction of precise solutions to boundary value problems in practical applications allows us to identify the properties and phenomena that have been omitted when using various approximate approaches. These include approximate analytical and numerical methods. Thus, the recently developed using of block element method [1] allowed us to identify conditions for occurrence of certain types of earthquakes [2,3]. The same method it made it possible to detect the existence of a new type of cracks that complement the Griffiths cracks [4]. A huge number of papers have been devoted to the study of boundary value problems for the Lame equation, containing both analytical and numerical studies performed in more than a century and a half. Not all publications in this area can be covered.

Note those of them where it was possible to build accurate analytical solutions of some types of boundary value problems for Lame vector equations in non-classical domains. We will omit from consideration numerous works devoted to boundary value problems in a half-space and a layered environment, where the Fourier transform solves the problem. In spher-

Бабешко Владимир Андреевич, академик РАН, д-р физ.-мат. наук, заведующий кафедрой математического моделирования Кубанского государственного университета, директор Научно-исследовательского центра прогнозирования и предупреждения геоэкологических и техногенных катастроф Кубанского государственного университета, заведующий лабораторией Южного федерального университета; e-mail: babeshko41@mail.ru.

Евдокимова Ольга Владимировна, д-р физ.-мат. наук, главный научный сотрудник Южного научного центра РАН; e-mail: evdokimova.olga@mail.ru.

Бабешко Ольга Мефодиевна, д-р физ.-мат. наук, главный научный сотрудник научно-исследовательского центра прогнозирования и предупреждения геоэкологических и техногенных катастроф Кубанского государственного университета; e-mail: babeshko49@mail.ru.

This work was supported by the Russian Foundation for Basic Research (projects 19-41-230003, 19-41-230004, 19-48-230014, 18-08-00465, 18-01-00384, 18-05-80008), the GZ UNC RAS reg. 01201354241 (project 00-20-13), Ministry of the science Russian Federation (project FZEN-2020-0022) charged on 2020 year.

ical areas, we should note the works devoted to the construction of eigenvector functions [5]. This approach has been developed for use in cylindrical, elliptical, wedge-shaped, and conic regions [6, 7].

In this paper, we develop an approach based on the possibility of decomposing the solution of the Lame vector equation into potential and vortex components, each of which is described in the dynamic case by solutions of the Helmholtz equation [8]. The difficulty of applying this method to boundary value problems in non-classical domains is explained by the difficulty of satisfying boundary conditions. Therefore, in the works [8–10] in which important relations of representation of solutions of vector boundary value problems by scalars are constructed, the solutions are constructed only for the half-space. The block element method for the first time allowed us to construct an exact solution to a complex plane boundary value problem for the Lame equations in the first quadrant by constructing a series of scalar boundary value problems for the Helmholtz equation. For comparison, the exact solution of the vector plane dynamic boundary value problem for the Lame equation in the first quadrant is given, constructed by direct application of the block element method to the Lame vector equation. The latter, obtained for the first time, is quite difficult to study and apply.

1. Basic equation

Let's consider a plane boundary value problem of the second kind for a system of Lamé equations, set in the first quadrant under harmonic influences on the boundary. Previously, it was not possible to get an exact solution to this problem, but the block element method in this paper makes it possible to do this in the form of packed vector block elements.

In the first quadrant, Lame's dynamic equations, after excluding the term $\exp(-i\omega t)$, have the form

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_1}+\mu\Delta u_1+k^2u_1=0,\qquad \theta=\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2},\quad k^2=\rho\omega^2, \tag{1.1}$$

$$(\lambda+\mu)\frac{\partial\theta}{\partial x_2}+\mu\Delta u_2+k^2u_2=0,$$

$$x_1,x_2\in\Omega,\quad \Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}.$$

Here $u_n(x_1,x_2)$ are the components of the displacement vectors at the point x_1, x_2 Ω, – the area of the first quadrant $x_1\geq 0$, $x_2\geq 0$, λ, μ – Lame parameters, ρ – the density of the material of the deformable body, ω – the frequency of external harmonic influences at the boundary, set by the complex function $\exp(-i\omega t)$, where $t$is the time. In a problem of the first kind, the stress values at the boundaries of a square are denoted on the abscissa axis by the functions $X_{x_2x_1}(x_1,0)$, $Y_{x_2x_1}(x_1,0)$, and, $X_{x_1x_2}(0,x_2)$, $Y_{x_1x_2}(0,x_2)$ – on the ordinate axis. Normal to the boundary stresses are indicated by the symbol $X$and tangents – Y. In a problem of the second kind, the components of the displacement vectors $u_1(x_1,0)$, $u_2(x_1,0)$ and $u_1(0,x_2)$, $u_2(0,x_2)$ are set at the boundary of the first quadrant. The displacement u_1 is directed along the normal line to the border.

2. Solving a vector boundary value problem using the direct block element method

After plunging the boundary value problem into the topological space of slowly growing generalized functions [1], applying Fourier transform $\mathbf{F}_2(\alpha_1,\alpha_2)$ operators and an external algebra algorithm, we arrive at a system of functional equations that have the form in the matrix representation

$$\mathbf{B}(\alpha_1,\alpha_2)\,\mathbf{U}(\alpha_1,\alpha_2)=\boldsymbol{\omega},$$

$$\mathbf{B}(\alpha_1,\alpha_2)=\|b_{mn}\|,$$

$$b_{11}=(\lambda+2\mu)\,\alpha_1^2+\mu\alpha_2^2-k^2,$$

$$b_{12}=b_{21}=(\lambda+\mu)\,\alpha_1\alpha_2,$$

$$b_{22}=(\lambda+2\mu)\,\alpha_2^2+\mu\alpha_1^2-k^2,$$

$$\mathbf{U}(\alpha_1,\alpha_2)=\{U_1(\alpha_1,\alpha_2),U_2(\alpha_1,\alpha_2)\}.$$

Here the notation of the Fourier transform is accepted

$$U_n(\alpha_1,\alpha_2)=\mathbf{F}_2(\alpha_1,\alpha_2)\,u_n(x_1,x_2)=\int_{-\infty}^{\infty}\int u_n(x_1,x_2)\,e^{i\langle\boldsymbol{\alpha}\mathbf{x}\rangle}dx_1dx_2,$$

$$\langle\boldsymbol{\alpha}\mathbf{x}\rangle=\alpha_1x_1+\alpha_2x_2,$$

$$\boldsymbol{\omega}(\alpha_1,\alpha_2)=\{\omega_1,\omega_2\},$$

$$\omega_1(\alpha_1,\alpha_2)=\omega_{11}(0,\alpha_2)+\omega_{12}(\alpha_1,0),$$

$$\omega_2(\alpha_1,\alpha_2)=\omega_{21}(\alpha_1,0)+\omega_{22}(0,\alpha_2).$$

The components of an external form $\boldsymbol{\omega}(\alpha_1,\alpha_2)$ vector have notation

$$\omega_{11}(0,\alpha_2) = -\sigma_{x_1x_1}(0,\alpha_2) + i\left[(\lambda+2\mu)\alpha_1 U_1(0,\alpha_2) + \mu\alpha_2 U_2(0,\alpha_2)\right],$$

$$\omega_{22}(0,\alpha_2) = -\tau_{x_1x_2}(0,\alpha_2) + i\left[\mu\alpha_1 U_2(0,\alpha_2) + \lambda\alpha_2 U_1(0,\alpha_2)\right],$$

$$\omega_{12}(\alpha_1,0) = -\tau_{x_2x_1}(\alpha_1,0) + i\left[\mu\alpha_2 U_1(\alpha_1,0) + \lambda\alpha_1 U_2(\alpha_1,0)\right],$$

$$\omega_{21}(\alpha_1,0) = -\sigma_{x_2x_2}(\alpha_1,0) + i\left[(\lambda+2\mu)\alpha_2 U_2(\alpha_1,0) + \mu\alpha_1 U_2(\alpha_1,0)\right].$$

Here $\sigma_{x_1x_1}(0,\alpha_2)$, $\sigma_{x_2x_2}(\alpha_1,0)$ are the Fourier transform of the normal $X_{x_1x_1}(0,x_2)$, $X_{x_2x_2}(x_1,0)$ and $Y_{x_2x_1}(x_1,0)$, $Y_{x_1x_2}(0,x_2)$ – tangential components of the stresses on the boundary of the quadrant. The determinant of the functional equation has the form

$$\det B = B_0\left[(\lambda+\mu)\left(\alpha_1^2+\alpha_2^2\right)+B_0\right],$$
$$B_0 = \mu(\alpha_1^2+\alpha_2^2) - k^2.$$

Zeros for each parameter of the determinant are represented in the form

$$\alpha_{11+} = i\sqrt{\alpha_2^2 - (\lambda+2\mu)^{-1}k^2},$$
$$\alpha_{12+} = i\sqrt{\alpha_2^2 - \mu^{-1}k^2},$$
$$\alpha_{21+} = i\sqrt{\alpha_1^2 - (\lambda+2\mu)^{-1}k^2},$$
$$\alpha_{22+} = i\sqrt{\alpha_1^2 - \mu^{-1}k^2}.$$

To study the boundary value problem using the block element method at the stage of external analysis, it is necessary to perform differential factorization of the matrix function and perform the operation of constructing an automorphism. For this purpose, taking into account the properties of the space of slowly growing generalized functions, we construct a representation of the solution of the matrix functional equation in the form

$$\mathbf{U}(\alpha_1,\alpha_2) = \mathbf{B}^{-1}(\alpha_1,\alpha_2)\,\boldsymbol{\omega}(\alpha_1,\alpha_2), \quad (2.1)$$

$$\mathbf{B}^{-1} = \frac{1}{\det\mathbf{B}}\begin{Vmatrix}(\lambda+\mu)\alpha_2^2+B_0 & -(\lambda+\mu)\alpha_1\alpha_2\\ -(\lambda+\mu)\alpha_1\alpha_2 & (\lambda+\mu)\alpha_1^2+B_0\end{Vmatrix}.$$

In the future, you need to perform an automorphism, which consists in the requirement of turning the solution vector to zero outside the carrier

$$\mathbf{F}^{-1}(x_1,x_2)\,\mathbf{B}^{-1}(\alpha_1,\alpha_2) \times \boldsymbol{\omega}(\alpha_1,\alpha_2) = 0, \quad (2.2)$$
$$x_1,x_2 \notin \Omega,$$

$\mathbf{F}^{-1}(x_1,x_2)$ – inverse Fourier transform operator

For the correct implementation of automorphism, a differential factorization of the matrix function is performed and the necessary selection of components of the vector of the external form is made.

Matrix functions have the form

$$\mathbf{R}_{mn} = (\alpha_m - \alpha_{mn+})^{-1}\begin{Vmatrix}\alpha_m - \alpha_{mn+} & 0\\ 1 & C_{mn}\end{Vmatrix},$$

$$C_{11}(\alpha_{11+},\alpha_2) = -\frac{b_{11}(\alpha_{11+},\alpha_2)}{b_{21}(\alpha_{11+},\alpha_2)},$$
$$C_{12}(\alpha_{12+},\alpha_2) = -\frac{b_{11}(\alpha_{12+},\alpha_2)}{b_{21}(\alpha_{12+},\alpha_2)},$$
$$C_{21}(\alpha_1,\alpha_{21+}) = -\frac{b_{11}(\alpha_1,\alpha_{21+})}{b_{21}(\alpha_1,\alpha_{21+})},$$
$$C_{22}(\alpha_1,\alpha_{22+}) = -\frac{b_{11}(\alpha_1,\alpha_{22+})}{b_{21}(\alpha_1,\alpha_{22+})}.$$

In the problem of elasticity theory of the second kind, displacement vectors are defined at the boundaries of a square. As a result of fulfilling the automorphism requirement, a system of pseudo-differential equations is constructed, which takes the form for the components of the vector of the external form

$$-\sigma_{x_1x_1}(0,\alpha_2) - \tau_{x_2x_1}(\alpha_{11+},0) + i\left[(\lambda+2\mu)\alpha_{11+}U_1(0,\alpha_2) + \mu\alpha_2 U_2(0,\alpha_2)\right] + i\left[\mu\alpha_2 U_1(\alpha_{11+},0) + \lambda\alpha_{11+}U_2(\alpha_{11+},0)\right] = 0,$$

$$-\sigma_{x_1x_1}(0,\alpha_{21+}) - \tau_{x_2x_1}(\alpha_1,0) + i\left[(\lambda+2\mu)\alpha_1 U_1(0,\alpha_{21+}) + \mu\alpha_{21+}U_2(0,\alpha_{21+})\right] + i\left[\mu\alpha_{21+}U_1(\alpha_1,0) + \lambda\alpha_1 U_2(\alpha_1,0)\right] = 0,$$

$$-\sigma_{x_2x_2}(\alpha_{12+},0) - \tau_{x_1x_2}(0,\alpha_2) + i\left[(\lambda+2\mu)\alpha_2 U_2(\alpha_{12+},0) + \mu\alpha_{12+}U_2(\alpha_{12+},0)\right] + i\left[\mu\alpha_{12+}U_2(0,\alpha_2) + \lambda\alpha_2 U_1(0,\alpha_2)\right] = 0,$$

$$-\sigma_{x_2x_2}(\alpha_1,0)-\tau_{x_1x_2}(0,\alpha_{22+})$$
$$+i\left[(\lambda+2\mu)\,\alpha_{22+}U_2(\alpha_1,0)+\mu\alpha_1U_2(\alpha_1,0)\right]$$
$$+i\left[\mu\alpha_1U_2(0,\alpha_{22+})+\lambda\alpha_{22+}U_1(0,\alpha_{22+})\right]=0.$$

In the case of solving a boundary value problem of the second kind, the displacement vectors are given at the boundaries of the square, and the stress components are unknown in pseudo-differential equations. The system of pseudo-differential equations, taking into account the requirement (2.2), allows a solution in vector form. As a result, external forms are represented as

$$\omega_1(\alpha_1,\alpha_2)=\left[s_1(\alpha_1,\alpha_2)-s_1(\alpha_{11+},\alpha_2)\right]-\left[s_1(\alpha_1,\alpha_{21+})-s_1(\alpha_{11+},\alpha_{21+})\right],$$

$$\omega_2(\alpha_1,\alpha_2)=\left[s_2(\alpha_1,\alpha_2)-s_2(\alpha_{12+},\alpha_2)\right]-\left[s_2(\alpha_1,\alpha_{22+})-s_2(\alpha_{12+},\alpha_{22+})\right].$$

Here it is marked

$$s_1(\alpha_1,\alpha_2)=i\left[(\lambda+2\mu)\,\alpha_1U_1(0,\alpha_2)+\mu\alpha_2U_2(0,\alpha_2)\right]+i\left[\mu\alpha_2U_1(\alpha_1,0)+\lambda\alpha_1U_2(\alpha_1,0)\right],$$

$$s_2(\alpha_1,\alpha_2)=i\left[\mu\alpha_1U_2(0,\alpha_2)+\lambda\alpha_2U_1(0,\alpha_2)\right]+i\left[(\lambda+2\mu)\,\alpha_2U_2(\alpha_1,0)+\mu\alpha_1U_1(\alpha_1,0)\right],$$

$$s_1(\alpha_1,\alpha_2)-s_1(\alpha_{11+},\alpha_2)=i(\lambda+2\mu)\left[\alpha_1U_1(0,\alpha_2)-\alpha_{11+}U_1(0,\alpha_2)\right]+i\mu\left[\alpha_2U_1(\alpha_1,0)-\alpha_2U_1(\alpha_{11+},0)\right]+\lambda\left[\alpha_1U_2(\alpha_1,0)-\alpha_{11+}U_2(\alpha_{11+},0)\right],$$

$$s_2(\alpha_1,\alpha_{22+})-s_2(\alpha_{12+},\alpha_{22+})=\mu\left[\alpha_1U_2(0,\alpha_{22+})-\alpha_{12+}U_2(0,\alpha_{22+})\right]+i\mu\left[\alpha_1U_1(\alpha_1,0)-\alpha_{12+}U_1(\alpha_{12+},0)\right]+(\lambda+2\mu)\left[\alpha_{22+}U_2(\alpha_1,0)-\alpha_{22+}U_2(\alpha_{12+},0)\right].$$

The constructed solutions include components of the displacement vector set at the quadrant boundary. As a result of adding the constructed solutions of pseudo-differential equations to the right parts of the external forms in (2.1), we obtain a representation of a packed vector block element, which is an exact solution of the boundary value problem for the Lame equations in the first quadrant, in the form

$$\mathbf{u}(x_1,x_2)=\mathbf{F}^{-1}(x_1,x_2)\,\mathbf{B}^{-1}(\alpha_1,\alpha_2)\,\boldsymbol{\omega}(\alpha_1,\alpha_2).\quad(2.3)$$

3. Solving a vector boundary value problem using solutions scalar boundary value problems

It was noticed long ago that the Lame equations, both in static and dynamic cases, have the property of representing the solution as the sum of potential and vortex components. It has been used in quite a large number of works, but only in simple areas – half-space, layered environment, and other areas obtained by representations of space transformation groups [5–10].

This is due to the fact that when decomposing the solution into potential and vortex components, it was not possible to perform such a decomposition under boundary conditions. According to the authors, this paper makes some progress in solving the problem of boundary conditions in this approach.

Following [8], we take the decomposition of the solution of the Lame equations in the following form

$$\begin{gathered}u_1(x_1,x_2)=\partial_1\phi(x_1,x_2)+\partial_2\psi(x_1,x_2),\\ u_2(x_1,x_2)=\partial_2\phi(x_1,x_2)-\partial_1\psi(x_1,x_2),\\ \partial_1=\frac{\partial}{\partial x_1},\quad\partial_2=\frac{\partial}{\partial x_2}.\end{gathered}\quad(3.1)$$

Here it is marked

$$\begin{gathered}\left(\Delta-p_1^2\right)\phi=0,\quad\left(\Delta-p_2^2\right)\psi=0,\\ p_1^2=k_1^2(\lambda+2\mu)^{-1},\quad p_2^2=k_1^2\mu^{-1},\\ \phi(x_1,0)=f_1(x_1,0),\\ \phi(0,x_2)=f_2(0,x_2),\\ \psi(x_1,0)=g_1(x_1,0),\\ \psi(0,x_2)=g_2(0,x_2).\end{gathered}\quad(3.2)$$

Functions f_m, g_m, $m=1,2$ that are arbitrary under boundary conditions, satisfying only the conditions of correctness of the statement boundary value problem. In particular, they can be taken from space slow-growing generalized functions that are searched for solutions to the boundary value problem in the domain Ω.

We consider the case of the second-kind Lame boundary value problem.

The following conditions are set on the coordinate axes: $u_n(x_1,0)$, $u_n(0,x_2)$, $n=1,2$.

Thus, for solutions of the Helmholtz equation, boundary conditions of the form are formed for $x_2 \to 0$

$$\begin{aligned}&\partial_1\phi(x_1,x_2)+\partial_2\psi(x_1,x_2)=u_1(x_1,0),\\&\partial_2\phi(x_1,x_2)-\partial_1\psi(x_1,x_2)=u_2(x_1,0).\end{aligned}\tag{3.3}$$

Similarly, when $x_1 \to 0$

$$\begin{aligned}&\partial_1\phi(x_1,x_2)+\partial_2\psi(x_1,x_2)=u_1(0,x_2),\\&\partial_2\phi(x_1,x_2)-\partial_1\psi(x_1,x_2)=u_2(0,x_2).\end{aligned}\tag{3.4}$$

The solution of the boundary value problem for Lame equations with boundary conditions (3.3), (3.4) requires the construction of solutions to boundary value problems for Helmholtz equations under arbitrary boundary conditions (3.2). This can be done using the block element method described in [1–4]. Examples of solving various boundary value problems using solutions of Helmholtz equations are available in [11–19]. The solution of the boundary value problem in the first quadrant, performed by the block element method, is available in [20]. In the Packed form in the first quadrant in the case of the Dirichlet boundary value problem the solutions have the form

$$\phi(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_1(\alpha_1,\alpha_2)}{(\alpha_1^2+\alpha_2^2-p_1^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1d\alpha_2,\tag{3.5}$$

$$\psi(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_2(\alpha_1,\alpha_2)}{(\alpha_1^2+\alpha_2^2-p_2^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1d\alpha_2,$$

$$\omega_1=\left[\frac{\alpha_1}{\alpha_{11+}}-1\right]\left\langle F_1(\alpha_2)-\frac{F_1(\alpha_{21+})\alpha_2}{\alpha_{21+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\left\langle F_2(\alpha_1)-\frac{\alpha_1F_2(\alpha_{11+})}{\alpha_{11+}}\right\rangle,$$

$$\omega_2=\left[\frac{\alpha_1}{\alpha_{12+}}-1\right]\left\langle G_1(\alpha_2)-\frac{G_1(\alpha_{22+})\alpha_2}{\alpha_{22+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{22+}}-1\right]\left\langle G_2(\alpha_1)-\frac{\alpha_1G_2(\alpha_{12+})}{\alpha_{12+}}\right\rangle,$$

$$\alpha_{11+}=i\sqrt{\alpha_2^2-p_1^2},\quad \alpha_{21+}=i\sqrt{\alpha_1^2-p_1^2},$$
$$\alpha_{12+}=i\sqrt{\alpha_2^2-p_2^2},\quad \alpha_{22+}=i\sqrt{\alpha_1^2-p_2^2}.$$

Sections for multi-valued functions are dictated by the requirement to perform automorphisms [1]. According to the construction, the properties (3.2) are valid for the given block elements. Using them, we introduce the following notations for solutions of the Helmholtz equations

$$\phi(x_1,x_2)\equiv\phi\left[x_1,x_2,f_1(\xi_1,0),f_2(0,\xi_2)\right]\to f_1(x_1,0),\quad 0<x_2\ll 1;$$

$$\phi(x_1,x_2)\equiv\phi\left[x_1,x_2,f_1(\xi_1,0),f_2(0,\xi_2)\right]\to f_2(0,x_2),\quad 0<x_1\ll 1;$$

$$\psi(x_1,x_2)\equiv\psi\left[x_1,x_2,g_1(\xi_1,0),g_2(0,\xi_2)\right]\to g_1(x_1,0),\quad 0<x_2\ll 1;$$

$$\psi(x_1,x_2)\equiv\psi\left[x_1,x_2,g_1(\xi_1,0),g_2(0,\xi_2)\right]\to g_2(0,x_2),\quad 0<x_1\ll 1.$$

These properties allow us to find a way to satisfy the boundary conditions of the boundary value problem for the Lame equations. Below, we will denote the integrals of a function by variables and first-order formulas and, respectively. So, there are representations

$$\begin{aligned}\partial_1^{(-1)}w(x_1,x_2)&=\int\limits_0^{x_1}w(\xi_1,x_2)d\xi_1,\\\partial_2^{(-1)}w(x_1,x_2)&=\int\limits_0^{x_2}w(x_1,\xi_2)d\xi_2.\end{aligned}\tag{3.6}$$

Obviousle

$$\partial_1\partial_1^{(-1)}w(x_1,x_2)=w(x_1,x_2),$$
$$\partial_2\partial_2^{(-1)}w(x_1,x_2)=w(x_1,x_2).$$

When solving The considered vector lame equation in the region representing the first quadrant with two intersecting boundaries, only one Packed block element of each boundary value problem for the Helmholtz equation is not sufficient to satisfy the boundary conditions. Each block element is a solution of the Helmholtz equations corresponding to the potential and vortex component of the solutions. This was sufficient for solving the boundary value problem for the Lame equation in a half-space with only one straight boundary, performed in [8]. It turned out that in the case of a polygon area, the number of block elements of scalar problems should be taken in the number of straight fragments that the boundary of the polygon area contains. Thus, to describe the solution Lama's equations in the first quadrant, which contains two straight fragments in the border, needed to take two block elements each potential and vortex components of the solution. Then the exact solution of the second boundary value problem for the Lame equation in the first quadrant is represented as

$$\begin{aligned} &u_1(x_1,x_2)\\ &= \partial_1\Big\langle \phi_1\Big[x_1,x_2,\frac{1}{2}\partial_1^{(-1)}u_1(\xi_1,0),\frac{1}{2}\partial_1^{(-1)}u_1(0,\xi_2)\\ &+\frac{1}{2}\partial_1^{(-1)}F(\xi_2)\Big]+\phi_2\Big[x_1,x_2,\frac{1}{2}\partial_2^{(-1)}u_2(\xi_1,0)\\ &\qquad+\frac{1}{2}\partial_1^{(-1)}D(x_1),\frac{1}{2}\partial_2^{-1}u_2(0,\xi_2)\Big]\Big\rangle\\ &\qquad+\partial_2\Big\langle\psi_1\Big[x_1,x_2,\frac{1}{2}\partial_2^{(-1)}u_1(\xi_1,0)\\ &\qquad+\frac{1}{2}\partial_1^{(-1)}C(x_1),\frac{1}{2}\partial_2^{(-1)}u_1(0,\xi_2)\Big]\\ &-\psi_2\Big[x_1,x_2,\frac{1}{2}\partial_1^{(-1)}u_2(\xi_1,0),\frac{1}{2}\partial_1^{(-1)}u_2(0,\xi_2)\\ &\qquad\qquad=\frac{1}{2}\partial_1^{(-1)}E(x_2)\Big]\Big\rangle; \end{aligned} \tag{3.7}$$

$$\begin{aligned} &u_2(x_1,x_2)\\ &= \partial_2\Big\langle \phi_1\Big[x_1,x_2,\frac{1}{2}\partial_1^{(-1)}u_1(\xi_1,0),\frac{1}{2}\partial_1^{-1}u_1(0,\xi_2)\\ &+\frac{1}{2}\partial_2^{(-1)}F(\xi_2)\Big]+\phi_2\Big[x_1,x_2,\frac{1}{2}\partial_2^{(-1)}u_2(\xi_1,0)\\ &\qquad+\frac{1}{2}\partial_2^{(-1)}D(x_1),\frac{1}{2}\partial_2^{-1}u_2(0,\xi_2)\Big]\Big\rangle\\ &\qquad-\partial_1\Big\langle\psi_1\Big[x_1,x_2,\frac{1}{2}\partial_2^{(-1)}u_1(\xi_1,0)\\ &\qquad+\frac{1}{2}\partial_2^{(-1)}C(x_1),\frac{1}{2}\partial_2^{(-1)}u_1(0,\xi_2)\Big]\\ &-\psi_2\Big[x_1,x_2,\frac{1}{2}\partial_1^{(-1)}u_2(\xi_1,0),\frac{1}{2}\partial_1^{(-1)}u_2(0,\xi_2)\\ &\qquad\qquad+\frac{1}{2}\partial_2^{(-1)}E(x_2)\Big]\Big\rangle. \end{aligned} \tag{3.8}$$

Here, the functions $C(x_2)$, $D(x_1)$, $E(x_2)$, $F(x_1)$ have the representation

$$C(x_1)=\partial_2\partial_1^{(-1)}u_1(x_1,0)-\partial_1\partial_2^{(-1)}u_1(x_1,0),$$

$$D(x_1)=\partial_2\partial_1^{(-1)}u_2(x_1,0)-\partial_1\partial_2^{(-1)}u_2(x_1,0),$$

$$E(x_2)=\partial_1\partial_2^{(-1)}u_2(0,x_2)-\partial_2\partial_1^{(-1)}u_2(0,x_2),$$

$$F(x_1)=\partial_1\partial_2^{(-1)}u_1(0,x_2)-\partial_2\partial_1^{(-1)}u_1\left(0,x_2\right).$$

It is easy to verify the validity of this statement by direct verification. Indeed, each packed block element, after applying the corresponding differential operators of the Helmholtz equations, takes the form

$$\phi(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\omega_1\left(\alpha_1,\alpha_2\right)\times e^{-i(\alpha_1x_1+\alpha_2x_2)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2,$$

$$\psi(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\omega_2\left(\alpha_1,\alpha_2\right)\times e^{-i(\alpha_1x_1+\alpha_2x_2)}\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha.$$

The right parts turn to zero due to the regularity of functions $\omega_1(\alpha_1,\alpha_2)$, $\omega_2(\alpha_1,\alpha_2)$, in the domain $\operatorname{Im}\alpha_1<0$, $\operatorname{Im}\alpha_2<0$ and decreasing exponential terms. The functions included in formulas (3.7) and (3.8) satisfy the first equation (3.2), and the functions – the second, only with different boundary conditions.

We show the implementation of the boundary conditions.

Let us limit ourselves to the consideration of the first boundary condition (3.7).

Using the above properties of packed block elements, we have the following chain of relations for fragments of solution (3.7):

$$\begin{aligned} &u_1(x_1,x_2)\\ &\to\partial_1\phi_1\Big[x_1,x_2,\frac{1}{2}\partial_1^{(-1)}u_1(\xi_1,0),\frac{1}{2}\partial_1^{-1}u_1(0,\xi_2)\\ &\qquad\qquad+\frac{1}{2}\partial_1^{(-1)}F_1(\xi_2)\Big]\\ &\qquad+\partial_2\psi_1\Big[x_1,x_2,\frac{1}{2}\partial_2^{(-1)}u_1(\xi_1,0)\\ &\qquad+\frac{1}{2}\partial_2^{(-1)}C_1(x_1),\frac{1}{2}\partial_2^{(-1)}u_1(0,\xi_2)\Big]\\ &\quad\to\frac{1}{2}u_1(x_1,0)+\frac{1}{2}u_1(x_1,0)=u_1(x_1,0); \end{aligned}$$

$$\partial_1\phi_2\Bigg[x_1, x_2, \frac{1}{2}\partial_2^{(-1)}u_2(\xi_1,0)$$
$$= \frac{1}{2}\partial_1^{(-1)}D_1(x_1), \frac{1}{2}\partial_2^{-1}u_2(0,\xi_2)\Bigg]$$
$$-\partial_2\psi_2\Bigg[x_1, x_2, \frac{1}{2}\partial_1^{(-1)}u_2(\xi_1,0), \frac{1}{2}\partial_1^{(-1)}u_2(0,\xi_2)$$
$$+\frac{1}{2}\partial_2^{(-1)}E_1(x_2)\Bigg]$$
$$\to \partial_1\frac{1}{2}\partial_2^{(-1)}u_2(x_1,0) + \frac{1}{2}D_1(x_1)$$
$$-\partial_2\frac{1}{2}\partial_1^{(-1)}u_2(x_1,0)$$
$$\to \partial_1\frac{1}{2}\partial_2^{(-1)}u_2(x_1,0) - \partial_2\frac{1}{2}\partial_1^{(-1)}u_2(x_1,0)$$
$$+\partial_2\frac{1}{2}\partial_1^{(-1)}u_2(x_1,0) - \partial_1\frac{1}{2}\partial_2^{(-1)}u_2(x_1,0) = 0.$$

The satisfaction of the other boundary conditions in (3.7), (3.8) is checked in exactly the same way.

Conclusion

In this paper, the same plane boundary value problem for The lame vector equation in the first quadrant is solved by two different approaches of the block element method. In the first case, the solution is constructed by direct application of the block element method to The lame vector boundary value problem. In the second case, a representation of the solution of the Lame equation using solutions of scalar Helmholtz equations is used. In both cases, exact solutions of boundary value problems are constructed for the first time. In the first case, where the matrix-function factorization operation was required at some stage, the solution is represented by a complex expression. In the second case, it is presented

Fairly simple solutions to scalar problems. Thus, it is shown that in cases where there is a representation of solutions of vector using solutions to scalar problems, it is advisable to use this approach.

References

1. Babeshko, V.A., Evdokimova, O.V., Babeshko, O.M., Evdokimov, V.S. Metod blochnogo elementa v razlozhenii resheniy slozhnykh zadach mekhaniki [Block element method in the decomposition of solutions to complex problems of mechanics]. *Doklady Akademii nauk* [Reports of the Academy of Sciences], 2020, vol. 495, no. 6, pp. 34–38. DOI: 10.31857/S2686740020060048 (In Russian)
2. Babeshko, V.A., Evdokimova, O.V., Babeshko, O.M. On the possibility of predicting some types of earthquake by a mechanical approach. *Acta Mechanica*, 2018, vol. 229, iss. 5, pp. 2163–2175. DOI: 10.1007/s00707-017-2092-0
3. Babeshko, V.A., Evdokimova, O.V., Babeshko, O.M. On a mechanical approach to the prediction of earthquakes during horizontal motion of litospheric plates. *Acta Mechanica*, 2018, vol. 229, pp. 4727–4739. DOI: 10.1007/s00707-018-2255-7
4. Babeshko, V.A., Evdokimova, O.V., Babeshko, O.M. A new type of cracks adding to Griffith–Irwin cracks. *Doklady Physics*, 2019, vol. 64, no. 2. pp. 102–105. DOI: 10.1134/S10283358191030042
5. Gelfand, I.M., Minlos, Z.A., Shapiro, Z.Ya. *Representations of the rotation group and the Lorentz group, their applications.* Fizmatgiz, Moscow, 1958. (In Russian)
6. Ulitko, A.F. *Method of eigenvector functions in spatial problems of elasticity theory.* Kiev, Naukova Dumka, 1979. (In Russian)
7. Grinchenko, V.T., Meleshko, V.V. *Harmonic vibrations and waves in elastic bodies.* Kiev, Naukova Dumka, 1981. (In Russian)
8. Nowacki, W. *Teoria sprezystosci.* Panstwowe Wydawnictwo Naukowe, Warsaw, 1970.
9. Nowacki, W. *Dynamiczne zagadnienia termosprezystosci.* Panstwowe Wydawnictwo Naukowe, Warsaw, 1966.
10. Nowacki W. *Efekty elektromagnetyczne w stalych cialach odksztalcalnych.* Panstwowe Wydawnictwo Naukowe, Warsaw, 1981.
11. Babich, V.M. On the Short-Wave Asymptotic Behaviour of the Green's Function for the Helmholtz Equation. *Mat. Sb. (N.S.)*, 1964, vol. 65, iss. 4, pp. 576–630.
12. Babich, V.M., Buldyrev, V.S. *Asymptotic methods in short-wavelength distraction theory.* Alpha Science International Ltd, 2009.
13. Mukhina I.V. Approximate reduction of the equations of the theory of elasticity and electrodynamics for inhomogeneous media to the Helmholtz equations. *Prikladnaya Matematika i Mekhanika*, 1972, vol. 36, iss. 4, pp. 667–671.
14. Molotkov, L.A. *Wave propagation in porous and cracked media, studied on the basis of effective biot models and layered media.* Nauka, Saint-Petersburg, 2001 (In Russian).
15. Tkacheva, L.A. Vibrations of a floating elastic plate due to periodic displacements of a bottom segment. *J. Appl. Mech. Tech. Phys.*, vol. 46, iss. 5, pp. 754–765.
16. Tkacheva, L.A. Plane problem of vibrations of an elastic floating plate under periodic external loading. *J. Appl. Mech. Tech. Phys.*, 2004, vol. 45, iss. 3, pp. 420–427.
17. Tkacheva, L.A. Behavior of a floating elastic plate during vibrations of a bottom segment. *J.*

Appl. Mech. Tech. Phys., 2005, vol. 46, iss. 2, pp. 230–238.
18. Tkacheva, L.A. Interaction of surface and flexural-gravity waves in ice cover with a vertical wall. *J. Appl. Mech. Tech. Phys.*, 2013, vol. 54, iss. 4, pp. 651–661.
19. Brekhovskikh, L.M. *Waves in layered media.* Academic Press, 1960.
20. Babeshko, V.A., Evdokimova, O.V., Babeshko, O.M. On the problem of acoustic and hydrodynamic properties of a medium occupying the area of a three-dimensional rectangular wedge. *J. Appl. Mech. Tech. Phys.*, 2019, vol. 60, iss. 6, pp. 90–96. DOI: 10.15372/PMTF20190610

Статья поступила 24 ноября 2020 г.