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ON THE PROPERTIES OF TOPOLOGICAL DISCRETIZATION OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS

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Abstract. This paper seems to show for the first time that Packed block elements used in solving a boundary value problem by the block element method are elements of a discrete topological space. Since the solutions to boundary value problems belong to a discrete topological space instead of equations, it is possible to obtain a solution in a new coordinate system without having to study the boundary value problem. The block element method used in problems of continuum mechanics, which has a connection with topology, can become more effective in applications if the General properties of topological spaces studied in detail are used more deeply. One of the important properties of topology is the existence of discrete topological spaces. Their characteristic property is that an element that represents the Union of any set of elements in a topological space belongs to a discrete topological space. Belonging of discrete block elements to a topological space means that it is possible to completely cover any area with a piecewise smooth boundary and, thus, an exact solution of the boundary problem in it. In topological space, continuous geometric transformations and transitions to new coordinate systems are possible. In this paper, it is shown that Packed block elements generated by the boundary value problem for the Helmholtz equation, are elements of a discrete topological space. Given that scalar solutions of the Helmholtz equation can describe solutions to a fairly wide set of vector boundary value problems, this property also applies to solutions of more complex boundary value problems. The constructions made for the homogeneous Helmholtz equation remain valid for the inhomogeneous one.

Keywords: boundary value problems, block element method, packed block elements, discrete topological spaces, Helmholtz equation.

Introduction

The block element method used in problems of continuum mechanics, which has a connection with topology, can become more effective in applications if the general properties of topological spaces studied in detail are used more deeply. One of the important properties of topology is the existence of discrete topological spaces. Their characteristic property is that an element that represents the union of any set of elements of a topological space belongs to a discrete topological space. Belonging of discrete block elements to a topological space means that it is possible to completely cover any region with a piecewise smooth boundary and, thus, to accurately solve the boundary problem in it. In topological space, continuous geometric transformations and transitions to new coordinate systems are possible.

In this paper, we show that the packed block elements generated by the boundary value problem for the Helmholtz equation, are elements of a discrete topological space. Given that scalar solutions of the Helmholtz equation can describe solutions to a fairly wide set of vector boundary value problems, this property also applies

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to solutions to more complex boundary value problems.

In [1, 2], the connection of block elements with various aspects of topology is discussed. In particular, it is shown that they are manifolds with an edge, they are characterized by homomorphisms, maps, and atlases. Aggregates of block elements are conjugated by constructing factor-topologies of factor-topological spaces, where the equivalence relations are assumed to be dictated by the requirements of mechanics boundary conditions under which deformable bodies are conjugated Theoretical materials related to the above questions are available in works [3–6]. However, theoretical constructions need to be implemented using concrete examples. In [7], an example of constructing a topological discretization in one of the topological spaces is briefly given. This paper provides a detailed and detailed presentation of the material. In this paper, the material is presented on the example of constructing a discrete topological space of solutions to the boundary value problem for the Helmholtz equation.

1. Construction of a discrete topological space

The objects of a topological space are the set of solutions of boundary value problems for the Helmholtz equation. The choice of these equations is not accidental. First, the boundary value problems for the Helmholtz equations describe the behavior of solutions to antiplane boundary value problems in elasticity theory. Secondly, the solutions of many more complex boundary value problems in the theory of elasticity, thermoelectromagnetodynamics, and mathematical physics are represented as the sum of solutions of boundary value problems for the Helmholtz equation or reduced to them [8–15]. This greatly simplifies the presentation of accurate solutions. In the near future, it will be shown for the first time that the block elements of the boundary value problem for the Helmholtz equation allow us to precisely satisfy the boundary conditions in the representation of solutions to complex boundary value problems in a number of non-classical domains that are more complex than those considered in [8–10]. Third, these boundary value problems are the simplest to illustrate new ideas, as in the present case. Let's perform the following constructions on the plane. Let us consider a homogeneous Helmholtz equation of the form in the entire plane

$$(\Delta + p^2)g = 0, \quad x_1, x_2 \in R^2 \equiv \Omega$$

There is no boundary value problem, and therefore no block element, but the equation in question is the most convenient for illustrating topological discretization. We begin to introduce a topological structure with the coarsest topology, the trivial one. We remind you that topologically, a block element represents the Cartesian product of the carrier and the solution of the boundary value problem on the carrier, represented as a packed block element. In our case, we have . The topological structure includes open sets in the carrier and solutions of the equation in packed block elements. The trivial topology includes the empty set and the reduced set . We extend the set by introducing some packed block elements in the considered set. We introduce four half-planes in a rectangular coordinate system with normals to the boundaries that coincide with the positive or negative coordinate half-axes. We introduce their notation $\Omega_1(|x_1| \leq \infty,\ x_2 > 0)$, $\Omega_2(|x_2| \leq \infty,\ x_1 < 0)$, $\Omega_3(|x_1| \leq \infty,\ x_2 < 0)$, $\Omega_4(|x_2| \leq \infty,\ x_1 > 0)$. In each domain, as a carrier, we solve boundary value problems for the Helmholtz equation with the following boundary conditions of the form

$$(\Delta + p^2)g_n = 0, \quad n = 1, 2, 3, 4,$$

$$\partial_1 = \frac{\partial}{\partial x_1}, \quad \partial_2 = \frac{\partial}{\partial x_2},$$

$$\partial_2 g_1(x_1, 0) = q_1(x_1), \quad g_1(x_1, x_2) \in \Omega_1$$

$$\partial_1 g_2(0, x_2) = q_2(x_2), \quad g_2(x_1, x_2) \in \Omega_2$$

$$\partial_2 g_3(x_1, 0) = q_1(x_1), \quad g_3(x_1, x_2) \in \Omega_3$$

$$\partial_1 g_4(0, x_2) = q_2(x_2), \quad g_4(x_1, x_2) \in \Omega_4$$

Here $q_n(x_n)$, $n = 1, 2$ are some smooth features.

In the following, we introduce the two-dimensional $\mathbf{F}_2$ and one-dimensional $\mathbf{F}_1$ Fourier transform operators, respectively, by putting

$$\mathbf{F}_2 \equiv \mathbf{F}_2(\alpha_1, \alpha_2), \quad \mathbf{F}_1 \equiv \mathbf{F}_1(\alpha_n),$$

$$Q(\alpha_n) = \mathbf{F}_1(\alpha_n) q(x_n) = \int_l q(x_n) \exp i\alpha_n x_n dx_n$$

$$G(\alpha_1, \alpha_2) = \mathbf{F}_2(\alpha_1, \alpha_2) g(x_1, x_2)$$
$$= \iint_\Omega g(x_1, x_2) \exp i(\alpha_1 x_1 + \alpha_2 x_2) dx_1 dx_2$$

Here l, Ω are the carriers of the integration functions.

We construct solutions to boundary value problems in the form of packed block elements. The latter have the form [1, 2, 7]

$$g_1(x_1,x_2) = \frac{1}{4\pi^2}\iint\limits_{R^2} \frac{Q_1(\alpha_1)}{(\alpha_1^2+\alpha_2^2-p^2)\alpha_{21+}}(\alpha_2-\alpha_{21+}) \times e^{-i(\alpha_1 x_1+\alpha_2 x_2)}d\alpha_1 d\alpha_2 \quad (1.1)$$

$$g_2(x_1,x_2) = \frac{1}{4\pi^2}\iint\limits_{R^2} \frac{Q_2(\alpha_2)}{(\alpha_1^2+\alpha_2^2-p^2)\alpha_{11-}}(\alpha_{11-}-\alpha_1) \times e^{-i(\alpha_1 x_1+\alpha_2 x_2)}d\alpha_1 d\alpha_2$$

$$g_3(x_1,x_2) = \frac{1}{4\pi^2}\iint\limits_{R^2} \frac{Q_1(\alpha_1)}{(\alpha_1^2+\alpha_2^2-p^2)\alpha_{21-}}(\alpha_{21-}-\alpha_2) \times e^{-i(\alpha_1 x_1+\alpha_2 x_2)}d\alpha_1 d\alpha_2 \quad (1.2)$$

$$g_4(x_1,x_2) = \frac{1}{4\pi^2}\iint\limits_{R^2} \frac{Q_2(\alpha_2)}{(\alpha_1^2+\alpha_2^2-p^2)\alpha_{11+}}(\alpha_1-\alpha_{11+}) \times e^{-i(\alpha_1 x_1+\alpha_2 x_2)}d\alpha_1 d\alpha_2$$

$$\alpha_{11+} = i\sqrt{\alpha_2^2-p^2}, \quad \alpha_{21+} = i\sqrt{\alpha_1^2-p^2}$$

$$\alpha_{11-} = -i\sqrt{\alpha_2^2-p^2}, \quad \alpha_{21-} = -i\sqrt{\alpha_1^2-p^2}$$

Here $Q_n(\alpha_s) = \mathbf{F}_1(\alpha_s)q_n(x_s)$.

Thus, the packed block elements are constructed in four mutually intersecting half-planes. They represent a block structure, the blocks of which have carriers-areas Ω_n, $n = 1, 2, 3, 4$. Consider the sets $\Omega_n \times g_n$, $n = 1, 2, 3, 4$. We introduce a topological structure in the constructed block structure. Let's call the internals of each constructed packed block element, and the carrier, that is, open sets, elements of a topological space. They must have the following properties: the union of any number of such elements and the intersection of a finite number of them must be an element of the topological space, that is, be a packed block element. In addition, the empty set and the entire set of elements are elements of a topological space. All of these requirements are met, except for one: the intersection of packed block elements must be a packed block element. Obviously, the carriers of packed block elements, that is, half-spaces with mutually perpendicular boundaries, have all four quadrants of the rectangular coordinate system as intersections. We introduce the following notation: $\Omega_5(x_1 > 0,\ x_2 > 0)$, $\Omega_6(x_1 < 0,\ x_2 > 0)$, $\Omega_7(x_1 < 0,\ x_2 < 0)$, $\Omega_8(x_1 > 0, x_2 < 0)$. Thus, in order to prove that the introduced topological structure $\Omega_n \times g_n,\ \Omega_{4+n} \times \phi_n$, $n = 1, \ldots, 4$ really forms a topological space consisting of carriers and packed block elements, it is necessary to show that the solutions of boundary value problems for the Helmholtz equation in each quadrant represent a packed block element. With respect to carriers, this question is solved thanks to the induced topology of Euclidean space. For block elements, any two adjacent block elements from the quadrants must combine and represent the packed block element of the half-space. Thus, using the block element method, it is necessary to construct solutions in the form of packed block elements in each quadrant of the following boundary value problems

$$(\Delta + p^2)\phi_n = 0,$$

$$\partial_2\phi_1(x_1,0) = q_{1+}(x_1), \quad \partial_1\phi_1(0,x_2) = q_{2+}(x_2), \quad \Omega_5(x_1 > 0,\ x_2 > 0)$$

$$\partial_2\phi_2(x_1,0) = q_{1-}(x_1), \quad \partial_1\phi_2(0,x_2) = q_{2+}(x_2), \quad \Omega_6(x_1 < 0,\ x_2 > 0)$$

$$\partial_2\phi_3(x_1,0) = q_{1-}(x_1), \quad \partial_1\phi_3(0,x_2) = q_{2-}(x_2), \quad \Omega_7(x_1 < 0,\ x_2 < 0)$$

$$\partial_2\phi_4(x_1,0) = q_{1+}(x_1), \quad \partial_1\phi_4(0,x_2) = q_{2-}(x_2), \quad \Omega_8(x_1 > 0,\ x_2 < 0)$$

Here $q_n(x_n)$, $n = 1, 2$ are some smooth features.

$$q_{1+}(x_1) = q_1(x_1), \quad x_1 \geqslant 0;$$

$$q_{2+}(x_2) = q_2(x_2), \quad x_2 \geqslant 0;$$

$$q_{1-}(x_1) = q_1(x_1), \quad x_1 \leqslant 0;$$

$$q_{2-}(x_2) = q_2(x_2), \quad x_2 \leqslant 0$$

Applying traditional block element methods, including the steps of external algebra, external analysis [1, 2, 7], we obtain four packed block elements in each quadrant

$$\phi_n(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_n\left(\alpha_1,\alpha_2\right)}{(\alpha_1^2+\alpha_2^2-p^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1 d\alpha_2$$

$$\omega_1=\left[\frac{\alpha_1}{\alpha_{11+}}-1\right]\times\left\langle Q_{2+}(\alpha_2)-\frac{\alpha_2 Q_{2+}(\alpha_{21+})}{\alpha_{21+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\left\langle Q_{1+}(\alpha_1)-\frac{\alpha_1 Q_{1+}(\alpha_{11+})}{\alpha_{11+}}\right\rangle$$

$$\omega_2=\left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\times\left\langle Q_{2+}(\alpha_2)-\frac{\alpha_2 Q_{2+}(\alpha_{21+})}{\alpha_{21+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\left\langle Q_{1-}(\alpha_1)-\frac{\alpha_1 Q_{1-}(\alpha_{11-})}{\alpha_{11-}}\right\rangle$$

$$\omega_3=\left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\times\left\langle Q_{2-}(\alpha_2)-\frac{\alpha_2 Q_{2-}(\alpha_{21-})}{\alpha_{21-}}\right\rangle+\left[1-\frac{\alpha_2}{\alpha_{21-}}\right]\left\langle Q_{1-}(\alpha_1)-\frac{\alpha_1 Q_{1-}(\alpha_{11-})}{\alpha_{11-}}\right\rangle$$

$$\omega_4=\left[\frac{\alpha_1}{\alpha_{11+}}-1\right]\times\left\langle Q_{2-}(\alpha_2)-\frac{\alpha_2 Q_{2-}(\alpha_{21-})}{\alpha_{21-}}\right\rangle+\left[1-\frac{\alpha_2}{\alpha_{21-}}\right]\left\langle Q_{1+}(\alpha_1)-\frac{\alpha_1 Q_{1+}(\alpha_{11+})}{\alpha_{11+}}\right\rangle$$

We make sure that the union of any two adjacent block elements that have carriers in the quadrants generates a block element in the form of a half-space. This operation is called the construction of a quotient topology, and the equivalence relations in this case consist in the equality of functions and their derivatives on the boundary. The named associations are the following objects . $\phi_1(x_1,x_2)\cup\phi_2(x_1,x_2)$, $\phi_2(x_1,x_2)\cup\phi_3(x_1,x_2)$, $\phi_3(x_1,x_2)\cup\phi_4(x_1,x_2)$, $\phi_4(x_1,x_2)\cup\phi_1(x_1,x_2)$.

Using the example of the first union, we show its transition to a packed block element for a half-space. We have

$$\phi_1(x_1,x_2)\cup\phi_2(x_1,x_2)=\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_1\left(\alpha_1,\alpha_2\right)}{(\alpha_1^2+\alpha_2^2-p^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1 d\alpha_2+ +\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_2\left(\alpha_1,\alpha_2\right)}{(\alpha_1^2+\alpha_2^2-p^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1 d\alpha_2= =\frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_1\left(\alpha_1,\alpha_2\right)+\omega_2\left(\alpha_1,\alpha_2\right)}{(\alpha_1^2+\alpha_2^2-p^2)}\times e^{-i(\alpha_1x_1+\alpha_2x_2)}d\alpha_1 d\alpha_2 \quad (1.3)$$

Hence

$$\omega_1+\omega_2=\left[\frac{\alpha_1}{\alpha_{11+}}-1\right]\times\left\langle Q_{2+}(\alpha_2)-\frac{Q_{2+}(\alpha_{21+})\alpha_2}{\alpha_{21+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\left\langle Q_{1+}(\alpha_1)-\frac{\alpha_1 Q_{1+}(\alpha_{11+})}{\alpha_{11+}}\right\rangle+ +\left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\left\langle Q_{2+}(\alpha_2)-\frac{Q_{2+}(\alpha_{21+})\alpha_2}{\alpha_{21+}}\right\rangle+\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\left\langle Q_{1-}(\alpha_1)-\frac{\alpha_1 Q_{1-}(\alpha_{11-})}{\alpha_{11-}}\right\rangle$$

in this relation, the expression

$$\frac{\alpha_1}{\alpha_{11+}},\quad\frac{\alpha_1}{\alpha_{11-}},\quad\frac{\alpha_2}{\alpha_{21+}},\quad\frac{\alpha_2}{\alpha_{21+}},\quad\frac{\alpha_2 Q_{2+}(\alpha_{21+})}{\alpha_{21+}},\quad\frac{\alpha_1 Q_{1+}(\alpha_{11+})}{\alpha_{11+}},\quad\frac{\alpha_1 Q_{1-}(\alpha_{11-})}{\alpha_{11-}}$$

they are called cut-offs. They perform functions that ensure the design of solutions to boundary value problems on carriers, that is, the conversion of the solution of the boundary value problem to zero outside the carrier. Their role pops up when calculating the inversions of Fourier transforms when obtaining values a packed block element in a Cartesian coordinate system.

Therefore, when the border between the block elements disappears, and operations with

Fourier transforms in external forms, they should be neglected, since the boundary disappears. The remaining ones are those that preserve the new boundaries of the packed block elements. With this in mind, discarding the unnecessary terms, we have

$$\omega_1+\omega_2=\left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\langle Q_{1+}(\alpha_1)\rangle + \left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\langle Q_{1-}(\alpha_1)\rangle = \left[\frac{\alpha_2}{\alpha_{21+}}-1\right]\langle Q_{1+}(\alpha_1)+Q_{1-}(\alpha_1)\rangle = \frac{\alpha_2-\alpha_{21+}}{\alpha_{21+}}Q_1(\alpha_1)$$

Adding this data to (1.3), we get a packed block element for the half-space (1.1), (1.2), and the union of block elements turns out to be a connected set. Similarly, combining the packed block elements of the second and third quadrants, we have

$$\omega_2+\omega_3=\left[1-\frac{\alpha_1}{\alpha_{11-}}\right] \times\left\langle Q_{2+}(\alpha_2)-\frac{\alpha_2 Q_{2+}(\alpha_{21+})}{\alpha_{21+}}\right\rangle + \left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\langle Q_{2-}(\alpha_2)\rangle = \left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\langle Q_{2+}(\alpha_2)\rangle = \frac{\alpha_{11-}-\alpha_1}{\alpha_{11-}}Q_2(\alpha_2)$$

Similarly, we get

$$\omega_3+\omega_4=\frac{\alpha_{21-}-\alpha_2}{\alpha_{21-}}Q_1(\alpha_1),$$

$$\omega_4+\omega_1=\frac{\alpha_1-\alpha_{11+}}{\alpha_{11+}}Q_2(\alpha_2)$$

There may be a question about combining the three quadrants. In this case, the external form takes the form $\omega_{13}=\omega_1+\omega_2+\omega_3$

From here we get a packed block element.

$$\phi_1(x_1,x_2)\cup\phi_2(x_1,x_2)\cup\phi_3(x_1,x_2) = \frac{1}{4\pi^2}\iint\limits_{R^2}\frac{\omega_{13}(\alpha_1,\alpha_2)}{(\alpha_1^2+\alpha_2^2-p^2)} \times e^{-i(\alpha_1 x_1+\alpha_2 x_2)}d\alpha_1 d\alpha_2$$

$$\omega_{13}(\alpha_1,\alpha_2)=\left[\frac{\alpha_2}{\alpha_{21+}}-1\right] \times\left\langle Q_{1+}(\alpha_1)-\frac{\alpha_1 Q_{1+}(\alpha_{11+})}{\alpha_{11+}}\right\rangle + \left[1-\frac{\alpha_1}{\alpha_{11-}}\right]\left\langle Q_{2-}(\alpha_2)-\frac{\alpha_2 Q_{2-}(\alpha_{21-})}{\alpha_{21-}}\right\rangle$$

Thus, it is proved that intersections of half-spaces are also packed block elements. It follows that the topological space will be defined if the elements in the form of half-planes are supplemented with block elements with carriers in the four quadrants of the coordinate system. They provide a stronger topology than half-planes. To construct a discrete topological space now, we need to use its definition. A discrete topological space is one in which any the union of space elements is an element of the same space. It is not necessary to prove that the elements of this space are packed block elements, with carriers in the four quadrants of the Cartesian coordinate system, having the strongest topology.

2. Properties of a discrete topological space

The closures of packed block elements of a discrete topological space are simultaneously manifolds with an edge and they have an atlas consisting of four single-card elements. The homomorphism for these elements is trivial, given by the carriers in the Cartesian coordinate system. The elements of a discrete topological space have no intersections, only the boundary sets of contact.

Having constructed a discrete topological space for boundary value problems of a simple differential equation, it is now possible to investigate and solve more complex boundary value problems using it.

The possibilities of discrete topological spaces of solutions of boundary value problems constructed in this way are quite wide. They allow, for example, to solve more complex boundary value problems based on solutions of simpler ones. Important is the property of topological spaces to allow continuous geometric transformations, transitions to new coordinate systems. Since not the equations, but the solutions of boundary value problems belong to the topological space, then, without requiring the solution of the boundary value problem in a new coordinate system, it is possible to obtain their solution in this coordinate system. This significantly expands the possibilities of the approach, since exact solutions are constructed, for example, in unbounded domains that are not available for investigation by numerical methods. Finally, the study based on the example of the Helmholtz equation makes it possible to transfer the results obtained in the scalar case to the vector one, using the approaches described in [11–15].

In the event that the original equation is inhomogeneous, that is

$$(\Delta + p^2)g(x_1, x_2) = m(x_1, x_2),$$

$$x_1, x_2 \in R^2 \equiv \Omega$$

all the described results and properties for solutions in new boundary value problems in the previously considered domains remain valid.

Conclusion

This paper, apparently for the first time, shows that the solutions of boundary value problems obtained by the block element method, presented in the form of packed block elements, make it possible to form a discrete topological space of solutions to the boundary value problem, which expand the possibilities of studying complex boundary value problems. In the near future, it will be shown how these results are transferred to vector boundary value problems of continuum mechanics.

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